INTRODUCTION IN THE USE OF THE CRANMER ABACUS

(DESIGNED FOR USE IN SPECIAL EDUCATION 443
METHODS AND MATERIALS IN TEACHING BLIND CHILDREN)

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BACKGROUND ON THE ABACUS AND ITS USE

Most methods of arithmetical calculations used by blind persons are cumbersome and inefficient. Taylor slates, cubarithum boards, and braille writers have all been successfully used by visually handicapped students; but these devices are awkward to carry and use. More significantly, calculating on them is extremely time consuming (Gissoni, 1963). Consequently, the mathematical performance of blind children falls 16-20 percent below the standards for sighted children (Nolan & Morris, 1964).

However, during the last decade, a new method of computation, the use of the Cranmer Abacus for the blind, has been developed, explored, and proved to be a significantly improved method for most visually handicapped students. The Cranmer Abacus is "....generally regarded by competent authorities as the best all-around computing device available to blind people" (Gissoni, 1965, p. 77).

The Cranmer Abacus is a modification of the Japanese abacus, or soroban, as it is called in Japanese. In Japan and other oriental nations the use of the abacus is very widespread. Its popularity in Japan is shown by the fact that it is used for 90 percent of all business arithmetic. The reasons for its popularity are numerous, and they indicate why its use among blind people has been very successful. The Japanese abacus is small, lightweight, and thus very easy to carry. It is inexpensive to buy and costs nothing to maintain. And most important, a skilled operator can achieve a degree of speed and accuracy that will enable him to outperform skilled operators of electric calculators (Gissoni, 1964).

The Cranmer Abacus was developed in 1962 by T. V. Cranmer, Director of the Division of Services for the blind in the Kentucky Department of Education. Mr. Cranmer was very interested in electronics and needed an efficient computational method to help solve his electronic formulas. He attempted to learn the use of the Japanese abacus; but, because he was blind, he found it impossible to work with the soroban counters (beads) which are free to move up and down on the wire columns with no resistance. The slightest touch of a counter caused it to move to an incorrect position. Since a blind person needs to read his answers tactually, this extremely free movement of the counters needed modification (Gissoni, 1964). Thus, Cranmer devised a foam and felt backing to hold the counters more stable; and he increased the length of the wire columns to make the abacus easier to handle and able to be read by touch (Lewis, 1969). Unfortunately, these modifications decrease the speed of computing on the abacus. Nevertheless, a blind operator of the Cranmer Abacus, while not as fast as a skilled operator of the Japanese style, is still able to make arithmetical calculations more quickly and more accurately than a blind student using another method (Nolan & Morris, 1964).

Since the development of the Cranmer Abacus, two separate tests have been written on its use. The first of these by Fred L. Gissoni was Using The Cranmer Abacus For The Blind, published by the American Printing House for the Blind in 1964. It was followed in 1966 by The Abacus Made Easy written by Mae E. Davidow,

and later published by the Printing House, also. I have studied both of these texts as well as a text on the use of the Soroban called The Japanese Abacus Explained by Yozo Yoshino.

Using the knowledge gained from these sources on how to use the abacus, I have written a detailed text for teaching its use to my own visually handicapped students. I have proceeded in a step-by-step manner through the processes of addition, multiplication, subtraction, division, fractional calculation, and the extraction of roots. Each major step is explained and exemplified, and appropriate exercises are given for the student to complete.

TABLE OF CONTENTS

ABOUT THE CRANMER ABACUS

The Cranmer Abacus is rectangular in shape measuring 3 x 6 inches and is 3/8 of an inch deep. The frame is plastic with a foam and felt backing. Attached to the frame are thirteen vertical wire rods or columns, each holding 5 counters or beads. Running the length of the frame is a horizontal bar which cuts across all thirteen columns and separates the top counter on each column from the remaining four. This bar is named the separation bar. On this bar, and also along the bottom of the frame there are single dots indicating the position of the columns. Also, from right to left, after every third column there is a line called a unit mark which marks the comma locations, and, in some cases, the decimal locations.

The abacus should generally be placed on a table or desk in front of the operator and left resting on that surface during its use. The single counters above the separation bar are called heaven counters and each of them equals five. The four counters below the separation bar on each column are called earth counters, and each of them is equal to one. When a counter is moved close to the bar, it takes on its assigned value (i.e. heaven = 5 each, earth = 1 each) when moved away from the bar it has no value. When we move a counter toward the bar, thus giving it value, we say we set it. On the other hand, when we move a counter away from the bar we say we clear it. When we say "set one left" or "clear one left" we mean to set or clear one on the column immediately to the left.

The column farthest right on the abacus is the units column, the next column left is the tens, then hundreds, and so on up to the thirteenth column which is trillions.

Before proceeding, be certain that you understand all of the terms explained above. The following checklist will help you:

column	set
counter	clear
separation bar	set left
unit mark	units column
heaven counter	tens column
earth counter	millions column

SETTING NUMBERS

We will begin our study of the use of the abacus by setting and clearing several numbers. Throughout this text you must pay careful attention to which finger is used for each movement. Doing so will help you attain the greatest speed possible. In most cases all setting and clearing will be done with the right hand. The left forefinger should rest on the column just to the left of the column being used - its function is to help you keep your position on the proper column.

Set 1 - Raise 1 earth counter on the units column with the thumb.

Clear 1 - Push down 1 earth counter on the units column with the right forefinger.

Set 2 - Raise 2 earth counters on the units column with the thumb.

Clear 2 - Push down 2 earth counters on the units column with the right forefinger.

Set 5 - Push down 1 heaven counter on the units column with the right forefinger.

Clear 5 - Push up 1 heaven counter on the units column with the right forefinger.

Set 6 - On the unit column, with one movement, push down 1 heaven counter with the forefinger and raise 1 earth counter with the thumb.

Clear 6 - On the unit column, push up 1 heaven counter with the forefinger; then push down 1 earth counter with the forefinger.

Set 8 - On the unit column, with one movement, push down 1 heaven counter with the forefinger and raise 3 earth counters with the thumb.

Clear 8 - On the unit column, push up 1 heaven counter with the forefinger; then push down 3 earth counters with the forefinger.

Set 10 - On the ten's column, raise 1 earth counter with the thumb.

Clear 10 - On the ten's column, push down 1 earth counter with the thumb.

Set 11 - On the ten's column, set 1 with the thumb.

On the unit's column, set 1 with the thumb.

Clear 11 - On the ten's column, clear 1 with the forefinger.

On the unit's column, clear 1 with the forefinger.

Set 37 - On the ten's column, set 3.

On the unit's column, set 7.

Clear 37 - On the ten's column, clear 3.

On the unit's column, clear 7.

Set 7,562 - On the thousand's column, set 7

On the hundred's column, set 5

On the ten's column, set 6

On the unit's column, set 2

Clear 7,562 - On the thousand's column, clear 7

On the hundred's column, clear 5

On the ten's column, clear 6

On the unit's column, clear 2

Practice setting many numbers (zip codes, telephone numbers, street numbers, class size, age birth years). Always set heaven counters with the right forefinger, and set earth counters with the right thumb. Clear both heaven and earth counters with the right forefinger. Use the left forefinger for positioning.

ADDITION

DIRECT ADDITION

Now that you have learned to set and clear numbers on the abacus, we are ready to begin the study of addition. Each number 1-9 can be added by one of three ways except the number 5 which has only two possible ways. The easiest method is that of Direct Addition, and we will study it first. Study carefully the examples below.

1 + 1 = 2

Raise an earth counter on the units column with the thumb
Raise another earth counter on the units column with the thumb;

The sum, 2, is indicated.

2 + 2 = 4

Raise two earth counters on the units column with the thumb
Raise two more earth counters on the units column with the thumb;

The sum, 4, is indicated.

1 + 5 = 6

Raise an earth counter on the units column with the thumb
Slide down a heaven counter on the units column with the forefinger;

The sum, 6, is indicated.

1 + 6 = 7

Raise an earth counter on the units column with the thumb

On the units column, with one movement, slide down a heaven counter with the forefinger and raise an earth counter with the thumb;

The sum, 7, is indicated.

2 + 7 = 9

Raise two earth counters on the units column with the thumb.

On the units column, with one movement, slide down a heaven counter with the forefinger and raise two earth counters with the thumb;

The sum, 9, is indicated.

When adding numbers of two or more digits, <u>always</u> start on the column farthest left and work to the right.

11 + 22 = 33

On the tens column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the tens column, raise two earth counters with the thumb.

On the units column, raise two earth counters with the thumb.

The sum, 33, is indicated.

52 + 27 = 79

On the tens column, slide down the heaven counter with the forefinger.

On the units column, raise two earth counters with the thumb.

On the tens column, raise two earth counters with the thumb.

On the units column, with one movement, slide down the heaven counter with the forefinger and raise two earth counters with the thumb.

The sum, 79, is indicated.

The examples above demonstrate the proper use of fingers in setting various numbers. The following problems will give you practice in direct addition. Careful attention should be paid to proper finger usage and left-to-right process.

1	1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	4	4	5	5
0	1	2	3	5	6	7	8	0	1	2	5	6	7	0	1	5	6	0	5	0	1
1	2	3	4	6	7	8	9	2	3	4	7	8	9	3	4	8	9	4	9	5	6

5	5	5	6	6	6	6	7	7	7	8	8	9	11	11	55	45	12	21
2	3	4	0	1	2	3	0	1	2	0	1	0	22	23	24	53	25	52
7	8	9	6	7	8	9	7	8	9	8	9	9	33	34	79	98	37	73

34	46	64	82	21	32	52	81	13	60	59	73	87	42
50	52	25	16	76	67	27	18	85	19	30	26	11	51
84	98	89	98	97	99	79	99	98	79	89	99	98	93

94	84	61	48	72	63	24	17	22	62	41	123	142
5	10	26	51	7	25	25	1	67	16	6	321	252
99	94	87	99	79	88	49	18	89	78	47	444	394

211	341	718	307	972	829	4263	7031	1234	2608	5862
627	556	120	671	17	160	5726	2663	3210	6231	4026
838	897	838	978	989	989	9989	9694	4444	8839	9888

8630	6135	36821	12345	36812	80324	78027	121212	326501
208	1562	12121	21503	62076	17165	10961	221112	162023
8838	7697	48942	33848	98888	97489	88988	342324	488524

1234567	87654321
1215121	12345678
2449688	99999999

SECRETS

Unfortunately, addition is not always direct. When we cannot add a number 1-9 directly, we must use a secret, or that is, a different process for adding that number. Each number 1-9 has two secrets except the number 5 which has only one. The secrets are listed below.

To Add	Secrets
1	Set 5, Clear 4
1	Clear 9, Set 1 left
2	Set 5, Clear 3
2	Clear 8, Set 1 left
3	Set 5, Clear 2
3	Clear 7, Set 1 left
4	Set 5, Clear 1
4	Clear 6, Set 1 left
5	Clear 5, Set 1 left
6	Set 1, Clear 5, Set 1 left
6	Clear 4, Set 1 left
7	Set 2, Clear 5, Set 1 left
7	Clear 3, Set 1 left
8	Set 3, Clear 5, Set 1 left
8	Clear 2, Set 1 left
9	Set 4, Clear 5, Set 1 left
9	Clear 1, Set 1 left

To gain skill and speed on the abacus, you must practice setting the secrets above until setting them becomes automatic. We will study the secrets for each number separately at first. On the pages ahead you will find many problems that you are to work for practice. Some of you will not need to work all of them, but there is no substitute for repetitive practice to develop skill and speed. The more problems you work, the better you will become.

SECRETS FOR ADDING ONE

Study and memorize the secrets for adding 1

Set 5, Clear 4

Clear 9, Set 1 left

For practice, add 1, plus 1, plus 1, plus 1, etc. until you feel confident in your use of the secrets. Follow the procedure outlined below to establish the proper finger usage.

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, slide down the heaven counter with the forefinger. Continue using the forefinger to clear the four earth counters. (Set 5, Clear 4)

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, raise one earth counter with the thumb.

On the units column, clear the heaven counter with the forefinger. Clear, the four earth counters with the forefinger and <u>simultaneously</u>, on the tens column, set an earth counter with the thumb. (Clear 9, Set 1 left).

On the units column set one earth counter with the thumb.

Continue on in this manner. When you reach 49, to add one, you must again clear 9, set 1 left. Because you cannot set 1 left directly on the tens column, you must therefore set 5, clear 4.

When you reach 99, to add one, you must again clear 9, set 1 left. Because you cannot set 1 left directly on the tens column, you must also clear 9, set 1 left on that column.

Continue practicing in this manner until the secrets for adding 1 become automatic reactions. For some of you this will mean adding until you reach 100. Others will have to reach a much higher sum. When you feel confident in your use of these secrets, work the following problems. Remember to use your fingers correctly and always to work from left to right.

111,111,111,111	444,444,444,444	666,666,666,666	999,999,999,999
111,111,111,111	111,111,111,111	111,111,111,111	111,111,111,111
222,222,222,222	555,555,555,555	777,777,777,777	1,111,111,111,110

123,456,789	65	76	34	47	81	42	48	64	24	44	74
111,111,111	11	11	11	11	11	11	11	11	11	11	11
234,567,900	76	87	45	58	92	53	59	75	35	55	85

46	54	91	79	96	39	98	94	69	29	19	49	89	237
11	11	11	11	11	11	11	11	11	11	11	11	11	111
57	65	102	90	107	50	109	105	80	40	30	60	100	348

643	474	461	489	194	9464	5497
111	111	111	111	111	1111	1111
754	585	572	600	305	10575	6608

SECRETS FOR ADDING TWO

Next we will study the secrets for adding 2.

Set 5, Clear 3

Clear 8, Set 1 left

For practice add 2, plus 2, plus 2, plus 2, etc. Follow the same pattern of finger usage as described for adding 1, plus 1, plus 1. Continue practicing until the secrets for adding two become automatic. Then work the following problems.

111,111,111,111	333,333,333,333	444,444,444,444	666,666,666,666
222,222,222,222	222,222,222,222	222,222,222,222	222,222,222,222
333,333,333,333	555,555,555,555	666,666,666,666	888,888,888,888

989,898,989,898	123,456,789	12	67	24	31	43	64	37
222,222,222,222	322,222,222	22	22	22	22	22	22	22
1,212,121,212,120	345,678,011	34	89	46	53	65	86	59

54	46	34	73	28	87	19	96	98	69	82	68	78	89	49	93	84	28
22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22
76	68	56	95	50	109	41	118	120	91	104	90	100	111	71	115	106	50

43	679	894	394	4893	9846	3749	69	74	35	79	98	974	678
22	222	222	222	2222	2222	2222	12	21	21	12	12	121	122
65	901	1116	616	7115	12068	5971	81	95	56	91	110	1095	800

SECRETS FOR ADDING THREE

Now study the secrets for adding 3.

Set 5, Clear 2

Clear 7, Set 1 left

For practice add 3, plus 3, plus 3, plus 3, etc. Follow the same pattern for finger usage as described in 1, plus 1, plus 1 etc. Continue practicing until the secrets for adding three become automatic. Then practice these problems.

222,222,222,222	444,444,444,444	777,777,777,777	686,868,686,868
333,333,333,333	333,333,333,333	333,333,333,333	333,333,333,333
555,555,555,555	777,777,777,777	1,111,111,111,110	1,020,202,020,201

123,456,789	16	51	26	54	43	24	34	64	41	32	23
333,333,333	33	33	33	33	33	33	33	33	33	33	33
456,790,122	49	84	59	87	76	57	67	97	74	65	56

634	243	81	96	89	97	78	18	98	67	76	59	85
333	333	33	33	33	33	33	33	33	33	33	33	33
967	576	114	129	122	130	111	51	131	100	109	92	118

698	789	63	84	97	28	73	84	39	27	374	893	297
333	333	33	33	33	33	33	33	33	33	333	333	333
1031	1122	96	117	130	61	106	117	72	60	707	1226	630

839	42	68	94	67	974	849	741
333	33	33	33	33	333	333	333
1172	75	101	127	100	1307	1182	1074

For a review of the secrets for adding 1, 2, and 3, add the number 123, 123 to itself several consecutive times to obtain these sums:

123123	1846845
246246	1969968
369369	2093091
492492	2216214
615615	2339337
738738	2462460
861861	2585583

984984	2708706
1108107	2831829
1231230	2954952
1354350	3078075
1477476	3201198
1600599	
1723722	

SECRETS FOR ADDING FOUR

Now study the secrets for adding 4.

Set 5, Clear 1

Clear 6, Set 1 left

For practice add 4, plus 4, plus 4, etc. Follow the same pattern for finger usage as described in 1, plus 1, plus 1, plus 1, etc. Continue practicing until the secrets for adding four become automatic. Then practice these problems.

333,333,333,333	555,555,555,555	777,777,777,777	828,282,828,282
444,444,444,444	444,444,444,444	444,444,444,444	444,444,444,444
777,777,777,777	999,999,999,999	1,222,222,222,221	1,272,727,272,726

123,456,789	55	150	51	15	53	31	24	40	121	243
444,444,444	44	444	44	44	44	44	44	44	444	444
567,901,233	99	594	95	59	97	75	68	84	565	687

134	65	77	86	95	69	97	78	68	57	867	776	698
444	44	44	44	44	44	44	44	44	44	444	444	444
578	109	121	130	139	113	141	122	112	101	1311	1220	1142

47	82	73	45	52	26	75	86	65	73	82	94	28	17
44	44	44	44	34	31	23	34	41	44	43	42	31	44
91	126	117	89	86	57	98	120	106	117	125	136	59	61

63	81	124	594	782	691	1869	3276
42	44	431	342	434	242	4444	4444
105	125	555	936	1216	933	6313	7720

SECRETS FOR ADDING FIVE

Now study the secret for adding 5.

Clear 5, Set 1 left

For practice, add 5, plus 5, plus 5, plus 5, etc. Follow the following procedure carefully to establish the proper finger usage:

5 On the units column, set the heaven counter with the forefinger.

+5 With one movement, clear the heaven counter with the forefinger (clear 5) and on the tens column, set 1 earth counter with the thumb. (Set 1 left).

+5 Repear Step #1, etc.

333,333,333,333	555,555,555,555	777,777,777,777	123,456,789
555,555,555,555	555,555,555,555	555,555,555,555	555,555,555
888,888,888,888	1,111,111,111,110	1,333,333,333,333	679,012,344

24	43	51	57	48	91	97	76	63	89	95	68	68	47
55	55	55	55	55	55	55	55	55	55	55	55	55	55
79	98	106	112	103	146	152	131	118	144	150	123	123	102

79	84	76	89	28	19	43	176	358	816	295
31	25	52	43	32	54	32	435	252	244	310
110	109	128	132	60	73	75	611	610	1060	605

SECRETS FOR ADDING SIX

Now study the secrets for adding 6.

Set 1, Clear 5, Set 1 left

Clear 4, Set 1 left

For practice, add 6, plus 6, plus 6, plus 6, etc. Follow the following procedure to establish proper finger usage.

6 - On the units column, with one movement, set the heaven counter with the forefinger and set 1 earth counter with the thumb (set 6)

+6 - On the units column, set 1 earth counter with the thumb. With one movement clear the heaven counter on the units column with the forefinger and set 1 earth counter on the tens column with the thumb. (Set 1, Clear 5, Set 1 left)

+6 - On the units column, with one movement, set the heaven counter with the forefinger and 1 earth counter with the thumb. (Set 6)

+6 - Set 1, Clear 5, Set 1 left.

+6 - With one movement, clear 4 earth counters on the units column with the forefinger and set 1 earth counter on the tens column with the thumb. (Clear 4, Set 1 left)

+6 - Set 6

+6 - Set 1, Clear 5, Set 1 left

+6 - Set 1, Clear 5, Set 1 left (To set 1 left here, you must set 5, clear 4 on the tens column.)

+6 - Clear 4, Set 1 left.

222,222,222,222	444,444,444,444	777,777,777,777	595,959,595,959
666,666,666,666	666,666,666,666	666,666,666,666	666,666,666,666
888,888,888,888	1,111,111,111,110	1,444,444,444,443	1,262,625,262,625

123,456,789	67	86	18	27	76	87	68	36	785	157
666,666,666	66	66	66	66	66	66	66	66	666	666
790,123,455	133	152	84	93	142	153	134	102	1451	823

558	687	14	42	44	49	19	94	43	34	499	293	124
666	666	66	66	66	66	66	66	66	66	666	666	666
1224	1353	80	108	110	115	85	160	109	100	1165	959	790

949	47	89	45	98	64	55	96	269	947	798	549
666	66	66	66	66	66	66	66	666	666	666	666
1615	113	155	111	164	130	121	162	935	1613	1464	1215

895	58	67	74	92	28	84	37	65	973	586	437	768
666	64	45	36	15	36	23	23	45	654	536	344	463
1561	122	112	110	107	64	107	60	110	1627	1122	781	1231

For a review of the secrets for adding, 1, 2, 3, 4, 5, and 6, add the number 123,456 to itself several consecutive times to obtain these sums:

123,456	1,851,840
246,912	1,975,296
370,368	2,098,752
493,824	2,222,208
617,280	2,345,664
740,736	2,469,120
864,192	2,592,576
987,648	2,716,032
1,111,104	2,839,488
1,234,560	2,962,944
1,358,016	3,086,400
1,481,472	3,209,856
1,604,928	3,333,312
1,728,384	

SECRETS FOR ADDING SEVEN

Now study the secrets for adding 7.

Set 2, Clear 5, Set 1 left

Clear 3, Set 1 left

For practice, add 7, plus 7, plus 7, plus 7, etc. Follow the same pattern for finger usage as described to add 6, plus 6, plus 6, etc.

7 - Set 7

+7 - Set 2, Clear 5, Set 1 left

+7 - Clear 3, Set 1 left

+7 - Set 7

+7 - Clear 3, Set 1 left

+7 - Set 2, Clear 5, Set 1 left

+7 - Set 7

+7 - Clear 3, Set 1 left (To set 1 left here, you must set 5, clear 4 on the tens column.)

333,333,333,333	666,666,666,666	437,896,549,786	65	57	61	15	56
777,777,777,777	777,777,777,777	777,777,777,777	77	77	77	77	77
1,111,111,111,110	1,444,444,444,443	1,215,674,327,563	142	134	138	92	133

165	567	676	576	775	675	14	42	81	19	94	48	39	394	948	429
777	777	777	777	777	777	77	77	77	77	77	77	77	777	777	777
942	1344	1453	1353	1552	1452	91	119	158	96	171	125	116	1171	1725	1206

893	938	47	68	59	96	78	98	58	985	687	548	349	937	82	79	24
777	777	77	77	77	77	77	77	77	777	777	777	777	777	34	43	37
1670	1715	124	145	136	173	155	175	135	1762	1464	1325	1126	1714	116	122	61

91	45	68	79	495	678	893	452
24	67	52	65	736	745	617	453
115	112	120	144	1231	1423	1510	905

SECRETS FOR ADDING EIGHT

Now study the secrets for adding 8.

Set 3, Clear 5, Set 1 left

Clear 2, Set 1 left

For practice, add 8, plus 8, plus 8, plus 8, etc. Follow the same pattern of finger usage as described in adding 6, plus 6, plus 6, etc.

8 - Set 8

+8 - Clear 2, Set 1 left

+8 - Set 3, Clear 5, Set 1 left

+8 - Clear 2, Set 1 left

+8 - Clear 2, Set 1 left

+8 - Set 8

+8 - Clear 2, Set 1 left (To set 1 left here, you must set 5, clear 4 on the tens column.)

333,333,333,333	666,666,666,666	777,777,777,777	796,853,794,678
888,888,888,888	888,888,888,888	888,888,888,888	888,888,888,888
1,222,222,222,221	1,555,555,555,554	1,666,666,666,665	1,685,742,683,566

15	61	16	56	516	655	565	165	561	656	665	40	34	74	83	29	12
88	88	88	88	888	888	888	888	888	888	888	88	88	88	88	88	88
103	149	104	144	1404	1543	1453	1053	1449	1544	1553	128	122	162	171	117	100

94	794	498	287	384	872	26	57	64	35	465	758	295	697	452	195
88	888	888	888	888	888	88	88	88	88	888	888	888	888	888	888
182	1682	1386	1175	1272	1760	114	145	152	123	1353	1646	1183	1585	1340	1083

674	85	48	26	98	63	18	37	534	795	486	582	873
888	37	86	45	27	48	56	37	686	316	734	738	264
1562	122	134	71	125	111	74	74	1220	1111	1220	1320	1137

SECRETS FOR ADDING NINE

Now study and memorize the secrets for adding 9.

Set 4, Clear 5, Set 1 left

Clear 1, Set 1 left

For practice, add 9, plus 9, plus 9, plus 9, etc. Continue following the pattern for finger usage described in adding 6, plus 6, plus 6, etc.

9 - Set 9

+9 - Clear 1, Set 1 left

+9 - Clear 1, Set 1 left

+9 - Clear 1, Set 1 left

+9 - Clear 1, Set 1 left

+9 - Set 4, Clear 5, Set 1 left (To set 1 left you must set 5, clear 4 on the tens column.)

333,333,333,333	555,555,555,555	777,777,777,777	459,678,972,185
999,999,999,999	999,999,999,999	999,999,999,999	999,999,999,999
1,333,333,333,332	1,555,555,555,554	1,777,777,777,776	1,459,678,972,184

15	51	35	58	55	535	525	455	295	185	573	681	497	598	281	349
99	99	99	99	99	999	999	999	999	999	999	999	999	999	999	999
114	150	134	157	154	1534	1524	1454	1294	1184	1572	1680	1496	1597	1280	1348

871	676	846	784	597	354	497	497	924	618	357	196	784	653	429
999	999	999	999	999	999	999	683	287	495	856	914	273	358	573
1870	1675	1845	1783	1596	1353	1496	1180	1211	1113	1213	1110	1057	1011	1002

367	5372	7394	3825	5728	2953	8129	7283
136	4629	2107	1175	2276	2047	2384	2263
503	10001	9501	5000	8004	5000	10513	9546

ADDITIONAL PRACTICE

For additional practice using all the secrets of addition, set the number 123,456,789 and add this number to itself many consecutive times to obtain the following sums.

123,456,789	2,839,506,147	5,555,555,505
246,913,578	2,962,962,936	5,679,012,294
370,370,367	3,086,419,725	5,802,469,083
493,827,156	3,209,876,514	5,925,925,872
617,283,945	3,333,333,303	6,049,382,661
740,740,734	3,456,790,092	6,172,839,450
864,197,523	3,580,246,881	6,296,296,239
987,654,312	3,703,703,670	6,419,753,028
1,111,111,101	3,827,160,459	6,543,209,817
1,234,567,890	3,950,617,248	6,666,666,606
1,358,024,679	4,074,074,037	6,790,123,395
1,481,481,468	4,197,530,826	6,913,570,184
1,604,938,257	4,320,987,615	7,037,036,973
1,728,395,046	4,444,444,404	7,160,493,762
1,851,851,835	4,567,901,193	7,283,950,551
1,975,308,624	4,691,357,982	7,407,407,340
2,098,765,413	4,814,814,771	7,530,864,129
2,222,222,202	4,938,271,560	7,654,320,918
2,345,678,991	5,061,728,349	7,777,777,707
2,469,135,780	5,185,185,138	7,901,234,496
2,592,592,569	5,308,641,927	8,024,691,285
2,716,049,358	5,432,098,716	8,148,148,074

8,271,604,863	9,259,259,175
8,395,061,652	9,382,715,964
8,518,518,441	9,506,172,753
8,641,975,230	9,629,629,542
8,765,432,019	9,753,086,331
8,888,888,808	9,876,543,120
9,012,345,579	9,999,999,909
9,135,802,386	

Another good exercise for practice in all the secrets of addition is to add:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$
$$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$$

$$1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 46$$
$$1 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 46$$

$$2 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 47$$
$$2 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 47$$

$$3 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 48$$
$$3 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 48$$

$$4 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 49$$
$$4 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 49$$

$$5 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 50$$
$$5 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 50$$

$$6 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 51$$
$$6 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 51$$

$$7 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 52$$
$$7 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 52$$

$$8 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 53$$
$$8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 53$$

$$9 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 54$$
$$9 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 54$$

Next add each of the following problems several times. ''ave someone read you the numbers as fast as you are able to add them, and try each time to complete the problem more quickly while still obtaining the correct sum.

16	31	68	75	87	189	795
21	69	84	16	31	327	829
85	87	31	91	75	685	763
93	34	92	84	29	432	304
27	29	76	30	72	709	210
68	76	43	69	19	375	693
41	87	54	54	64	640	542
76	94	47	73	39	496	317
39	32	68	49	77	70	619
22	539	31	32	34	694	5072
97		93	28	68	38	
85		86	601	45	4655	
34		25		640		
73		798				
777						

ADDITION OF DECIMALS

The same secrets are used in the addition of decimals as are used in the addition of whole numbers. There is, however, a specific procedure for determining proper placement of decimal numbers on the abacus that was not necessary for placing whole numbers.

1. First, find which addend (number being added) has the most decimal places.
2. If the largest number of decimal places is three or less, use the first unit mark from the right as the decimal place (thus such numbers as 47.2, 6.75, 6.734, 94.86, 345.1, would use the first unit mark as the decimal point.
3. If the largest number of decimal places if four, five, six, use the second unit mark from the right as the decimal point.

(Thus numbers such as 63.0064, 7.12876, 786,389576, 4.672571, would use the second unit mark as the decimal point.)

4. If the largest number of decimal places is seven, eight, or nine use the <u>third</u> unit mark from the right as the decimal point. (Thus, numbers such as 64.7443861, 1.758436612, 834.687435981 would use the third unit mark as the decimal point.)

Once you have determined which unit mark will be used as the decimal point, you need only remember to set each addend on the abacus so that the decimal point is positioned properly. The actual addition uses the same secrets which you have already mastered. Study the examples below.

Example #1

6.07 + 13.9 + 65.743

Largest number of decimal places is three - first unit mark is decimal point.

6.07	Set so that 6 is on column immediately to left of first unit mark.
13.9	Set so that 3 is on column immediately to left of first unit mark.
<u>65.743</u>	Set so that 5 is on column immediately to left of first unit mark.
85.713	Is the final sum - 5 is on colum immediately to left of first unit mark.

Example #2

7.5 + 879.4 + 62.46

Largest number of decimal places is two - <u>first</u> unit mark is decimal point.

7.5	Set so that 7 is on column immediately to left of first unit mark.
579.4	Set so that 9 is on column immediately to left of first unit mark.
<u>62.46</u>	Set so that 2 is on column immediately to left of first unit mark.
649.360	Is the sum - 9 is on column immediately to left of first unit mark. (The zero here has no significance since it does not serve as a place holder as in 4.07, 4.6009. Therefore it can either be recorded or omitted.

Example #3

84.035 + 78.7 + 98.6974 =

Largest number of decimal places is four - <u>second</u> unit mark is decimal point.

84.035	Is set so that 4 is on column immediately to left of second unit mark.
78.7	Is set so that 8 is on column immediately to left of second unit mark.
<u>98.6974</u>	Is set so that 8 is on column immediately to left of second unit mark.
261.4324	Is the sum - 1 is on column immediately to left of second unit mark.

Example #4

7.4798368 + .00976583 + .052 =

Largest number of decimal is 8 - <u>third</u> unit mark is decimal place.

7.4698368	Is set so that 7 is on column immediately to left of third unit mark.
.00976583	Is set so that 9 is on third column to right of third unit mark.
.052	Is set so that 5 is one second column to right of third unit mark.
7.53160263	Is the final sum - 7 is on column immediately to left of third unit mark.

Do the following problems for practice in adding decimals.

6.07	4.653	342.4	68.1	6.819	3.478
41.9	8.3	61.0	324.278	13.0	12.91
18.34	27.31	4.95	64.32	24.7	6.5876
66.31	491.0	408.35	12.6	44.519	22.9756
	531.263		469.298		

42.5913	14.9	64.71	1.07	.00286953	1.7
281.7	3.4785	389.6	6.7004	1.000640921	16.0473
65.834	126.35	10.0406	5.8069	22.8742009	9.0860764
9.6741	144.7285	464.3506	13.5773	23.877711351	26.8333764
399.7994					

MULTIPLICATION

As in multiplication on paper, a student using the abacus must have mastered the multiplication tables. The student who is just learning these tables can, of course, practice them on the abacus to attain perfection. He cannot, however, proceed further into this discussion of multiplication until he knows his tables completely.

In the multiplication problem, each part has a name; the number being multiplied is the multiplicand. The number doing the multiplying is the multiplier, and the answer is called the product. It is essential that you learn these names since they will be used throughout our explanation of multiplication.

Before we can begin to multiply, we must first set the multiplier and the multiplicand on the proper columns of the abacus. It is very important that they be located correctly so that the last digit of the product will always fall on the last column to the right of the abacus. Knowing which column is the last one in the product is necessary when the last digit is zero.

The multiplier is easy to locate. It is set on the columns farthest left on your abacus. To determine where to set the multiplicand, we first add the number of digits in the multiplier plus the number of digits in the multiplicand plus one. That sum is the number of the column (counting from the right) where the first digit of the multiplicand is set, the remaining digits being set in order to the right. For instance, in the problem 6 x 9 = ?, we set the

multiplier (6) on the column farthest left. We add the number of digits in the multiplier (1) plus the number of digits in the multiplicand (1) plus 1. The sum is 3. Thus the first digit of the multiplicand is set on the third column from the right. If the multiplier has two digits and the multiplicand three, then the first digit on the multiplicand is set on the 6th column from the right (2 + 3 + 1 = 6).

Practice setting these problems. Do not try to work them.

6 x 7 =	4 x 10 =	179 x 34 =	20 x 8000 =
4 x 8 =	17 x 90 =	1874 x 685 =	7 x 12050 =
9 x 3 =	28 x 4 =	204 x 691 =	18 x 4187 =
12 x 9 =	380 x 25 =	71 x 4 =	940 x 6800 =
47 x 2 =	650 x 1 =	80 x 704 =	2 x 4 =
6 x 28 =	27 x 38 =	67 x 400 =	6801 x 420 =
5 x 74 =	69 x 480 =	7 x 1700 =	800 x 6000 =

POSITION CONCEPT

When multiplying on the abacus, it is essential that we understand the position concept. We already know that there are thirteen columns on the abacus. A position consists of two adjacent columns. Which two columns comprise the first position depends upon the location of the multiplicand. The first position is always the first two columns after the multiplicand, regardless of where those two columns are on the abacus. The second position is always columns 2 + 3 to the right of the multiplicand. Notice that the positions overlap; column 2 is the last column of the first position, and the first

column of the <u>second position</u>. It follows that the <u>third position</u> is columns 3 and 4 to the right of the multiplicand, the <u>eighth position</u> is columns 8 and 9 and the fourteenth is columns 14 and 15.

Name which columns make up these positions.

Sixth position	- Columns 6 and 7 to right of multiplicand
Fourth position	- Columns 4 and 5 to right of multiplicand
Eleventh position	- Columns 11 and 12 to right of multiplicand
First position	- Columns 1 and 2 to right of multiplicand
Ninth position	- Columns 9 and 10 to right of multiplicand
Third position	- Columns 3 and 4 to right of multiplicand
Sixteenth position	- Columns 16 and 17 to right of multiplicand
Second position	- Columns 2 and 3 to right of multiplicand

ONE-DIGIT MULTIPLIER TIMES ONE-DIGIT MULTIPLICAND

We are now ready to consider multiplications having one-digit multipliers and one-digit multiplicands. Review how to determine where to set the multiplicand. The product of the first multiplication in any problem is set in the <u>first position</u> (i.e. columns 1 and 2 to the right of the multiplicand.) The following problems will all result in two-digit products, and thus will use both columns of the <u>first position</u>.

6 x 3 = 18	9 x 2 = 18	5 x 8 = 40	6 x 4 = 24
4 x 7 = 28	4 x 5 = 20	9 x 5 = 45	8 x 6 = 48

5 x 7 = 35	3 x 7 = 21	6 x 8 = 48	6 x 5 = 30
8 x 4 = 32	6 x 6 = 36	9 x 9 - 81	7 x 8 = 56

In the following exercises, the products are one-digit numbers. One-digit products are always set on the <u>second column</u> of the position. Work these problems.

2 x 3 = 6	2 x 4 = 8	2 x 2 = 4	9 x 1 = 9	3 x 2 = 6
1 x 5 = 5	9 x 1 = 9	1 x 8 = 8	3 x 3 = 9	4 x 1 = 4
7 x 1 = 7	3 x 3 = 9	5 x 1 = 5	1 x 1 = 1	2 x 2 = 4

ONE-DIGIT MULTIPLIER TIMES TWO-DIGIT MULTIPLICAND

Next we will consider problems having a one-digit multiplier and a two-digit multiplicand. In multiplying this type of problem we first determine the position of the multiplier and multiplicand. The multiplier, as always, is set as far left on the abacus as possible. Again we count the number of digits in the multiplier, add to it the number of digits in the multiplicand, plus one. The total, in this case, is 4. On the fourth column from the right edge of the abacus, set the first digit of the multiplicand, and set the rest of the digits in order to the right.

Study carefully the following examples.

4 x 37 =

1. Multiply the <u>first</u> digit of the multiplier (in this case, the only digit) times the <u>last</u> digit of the multiplicand, i.e. 4 x 7.
2. Set the product (28) in the first position to the right of

the multiplicand.

3. Clear the last digit of the multiplicand.
4. Multiply the first digit of the multiplier (4) times the remaining digit of the multiplicand (3).
5. Set the product (12) in the first position to the right of the remaining multiplicand. (Remember that the first position is always the first two columns to the right of whatever part of the multiplicand remains. In this case, the 1 is set in the column immediately to the right of the 3 of the multiplicand. The 2 of the product is set on the next column to the right which also is where the 2 of the first product (28) is set.
6. Clear the remaining multiplicand.
7. The product, 148, is now shown on the abacus.

Now try another problem of the same kind.

6 x 42 =

1. Multiply the <u>first</u> digit of the multiplier (6) times the <u>last</u> digits of the multiplicand (2).
2. Set the product (12) in the first position to the right of the multiplicand.
3. Clear the last digit of the multiplicand (2).
4. Multiply the first digit of the multiplier (6) times the remaining digit of the multiplicand (4).

5. Set the product (24) in the first position to the right of remaining multiplicand.

6. Clear the remaining multiplicand.

7. The product, 252, is now shown on the abacus.

Using the above procedure, work the following problems on your abacus.

3 x 9 = 288	4 x 39 = 156	8 x 59 = 472
8 x 72 = 576	9 x 26 = 234	6 x 62 = 372
5 x 86 = 430	7 x 47 = 329	4 x 87 = 348
7 x 58 = 406	2 x 98 = 196	2 x 69 = 138
4 x 54 = 216	8 x 46 = 368	6 x 58 = 348
9 x 85 = 765	5 x 39 = 195	9 x 43 = 387
3 x 68 = 204	7 x 84 = 588	8 x 67 = 536
7 x 92 = 644	4 x 68 = 272	6 x 78 = 468

2 x 47 =

To work this problem follow this procedure carefully.

1. Multiply the first digit of the multiplier (2) times the last digit of the multiplicand (7).

2. Set the product (14) in the first position to the right of the multiplicand.

3. Clear the last digit of the multiplicand (7).

4. Multiply the first digit of the multiplier (2) times the remaining digit of the multiplicand (4).

5. Set the product (8) in the second column of the first

position to the right of the remaining multiplicand. (Notice this is a one-digit product.)

6. Clear the remaining multiplicand.

7. The product, 94, is now shown on the abacus.

Using the above procedure, work the following problems.

4 x 26 = 104	8 x 19 = 152	7 x 14 = 98	6 x 16 = 96
2 x 39 = 78	3 x 37 = 111	4 x 23 = 92	3 x 27 = 81

4 x 62 =

Use the following procedure to work this problem 4 x 62:

1. Multiply the <u>first</u> digit of the multiplier (4) times the <u>last</u> digit of the multiplicand (2).
2. Set the product (8) in the <u>second column of the first position</u> to the right of the multiplicand. (one-digit product)
3. Clear the last digit of the multiplicand (2).
4. Multiply the first digit of the multiplier (4) times the remaining digit of the multiplicand (6).
5. Set the product (24) in the first position to the right of the remaining multiplicand.
6. Clear the remaining multiplicand.
7. The product, 248, is now shown on the abacus.

Using the above procedure, work the following problems.

3 x 93 = 279	6 x 8 1 = 486	4 x 92 = 368	3 x 72 = 216
4 x 41 = 124	2 x 63 = 126	7 x 61 = 427	2 x 84 = 168

3 x 21 =

Use the following procedure to work this problem.

1. Multiply the <u>first</u> digit of the multiplier (3) times the last digit of the multiplicand (1).
2. Set the product (3) in the <u>second column of the first position</u> to the right of the multiplicand.
3. Clear the last digit of the multiplicand (1).
4. Multiply the <u>first</u> digit of the multiplier (3) times the remaining digit of the multiplicand (2).
5. Set the product in the <u>second column of the first position</u> to the right of the multiplicand.
6. Clear the remaining multiplicand.
7. The product, 63, is now shown on the abacus.

Use the above procedure to work the following problems.

4 x 21 = 84	3 x 32 = 96	7 x 11 = 77	2 x 42 = 84
6 x 11 = 66	2 x 14 = 28	3 x 23 = 69	4 x 22 = 88

In the last three sets of practice exercises, you have been drilled in setting a one-digit product in the <u>second column of the position in use</u>. This is a <u>very important concept</u>. As we progress to different types of multiplication, this point will not be discussed again in detail. Still, it is absolutely necessary that you remember the concept and use it whenever a one-digit product occurs in any type of multiplication problem.

ONE-DIGIT MULTIPLIER TIMES THREE-DIGIT MULTIPLICANDS

Next, we will consider problems with one-digit multiplier, and three-digit multiplicands. The procedure is basically the same as that which we have already learned. Remember how to set the multiplicand. Follow this example carefully. Then work the following problems.

7 x 648

1. Multiply the first digit of the multiplier (7) times the last digit of the multiplicand (8).
2. Set the product (56) in the first position to the right of the multiplicand.
3. Clear the last digit of the multiplicand (8).
4. Multiply the first digit of the multiplier (7) times the <u>last</u> digit in the remaining multiplicand. (4)
5. Set the product (28) in the <u>first</u> position to the right of the remaining multiplicand.
6. Clear the last digit of the remaining multiplicand (4).
7. Multiply the first digit of the multiplier (7) times the last digit of the remaining multiplicand (6).
8. Set the product (42) in the first position to the right of the remaining multiplicand.
9. The final product, 4536, is now shown on the abacus.

9 x 382 = 3428	6 x 523 = 3138	7 x 836 = 5852
5 x 637 = 3185	2 x 968 = 1936	5 x 274 = 1370
4 x 857 = 3428	8 x 742 = 5936	6 x 792 = 4752
7 x 429 = 3003	9 x 475 = 4275	8 x 429 = 3432

6 x 381 = 2286	8 x 417 = 3336	3 x 287 = 861
4 x =	2 x 736 = 1472	9 x 134 = 1206
3 x 943 = 2829	4 x 629 = 2516	2 x 395 = 790
2 x 784 = 1568	7 x 314 = 2198	4 x 246 = 984
5 x 420 = 2100	9 x 141 = 1269	6 x 400 = 2400
8 x 960 = 7680	3 x 235 = 705	4 x 310 = 1240
3 x 420 = 1260	2 x 413 = 826	2 x 102 = 204
2 x 120 = 240	4 x 231 = 924	3 x 302 = 906
8 x 600 = 4800	3 x 603 = 1809	
7 x 408 = 2856	4 x 207 = 828	
3 x 905 = 2715	2 x 303 = 606	
5 x 804 = 4020	3 x 103 = 309	

ONE-DIGIT MULTIPLIER TIMES LARGER MULTIPLICANDS

The same procedure that you have been using in the last set of practice exercises is used whenever a one-digit multiplier is taken times a multiplicand with any number of digits. Work these problems which have four-five-six-seven-digit multiplicands.

8 x 6943 = 55544	6 x 9475 = 56850	7 x 4197 = 29379
7 x 4876 = 34132	5 x 8732 = 43660	6 x 1719 = 10314
4 x 3587 = 14348	3 x 9586 = 28758	3 x 3421 = 10263
9 x 5736 = 51624	8 x 6743 = 53944	2 x 4372 = 8744
4 x 2132 = 8528	6 x 3870 = 23220	4 x 3200 = 12800
3 x 3213 = 9639	4 x 6940 = 27760	3 x 6030 = 18090

2 x 4213 = 8426	2 x 3720 = 7440	2 x 7004 = 14008
3 x 4321 = 12063	5 x 2100 = 10100	3 x 2003 = 6009
6 x 3043 = 18258	6 x 87564 = 525384	2 x 38241 = 76482
7 x 6008 = 42056	9 x 69735 = 627615	6 x 30780 = 184680
5 x 4702 = 23510	5 x 61743 = 308715	9 x 40009 = 360081
3 x 2301 = 6903	4 x 28132 = 112528	3 x 30201 = 90603

9 x 465732 = 4191588	8 x 710003 = 5680024
4 x 826541 = 3306164	7 x 2735687 = 19149809
2 x 473214 = 946428	4 x 3241738 = 12966952
6 x 601073 = 3606438	6 x 8070120 = 48420720
2 x 2003104 = 4006208	3 x 4402601 = 13267803

9 x 246913578 = 2222222202

TWO-DIGIT MULTIPLIER TIMES ONE-DIGIT MULTIPLICAND

Now we are ready to multiply numbers having a two-digit multiplier. To multiply a two-digit multiplier times a one-digit multiplicand, follow this procedure carefully.

Remember:

Set the multiplicand in the correct location.

Use the proper finger and thumb motions.

Use the position concept correctly.

Study this example.

35 x 7

1. Set 3 on the column farthest left, set 5 on the column next

to it on the right. Set 7 on the 4th column from the right.

2. Multiply the first digit of the multiplier (3) times the last digit of the multiplicand (7).
3. Set the product (21) in the first position (first two columns to right of multiplicand). (7).
4. Multiply the 2nd number of the multiplier (5) times the last digit of the multiplicand. (7).
5. Set the product (35) in the second position (columns 2 and 3 to the right of the multiplicand.)
6. Clear the multiplicand (since it has now been multiplied by every digit in the multiplier).
7. The final product, 245, is now shown on the abacus.

Work the following problems using the procedures described above.

52 x 7 = 364	82 x 4 = 328	27 x 4 = 108
73 x 8 = 584	63 x 2 = 126	35 x 3 = 105
94 x 3 = 282	21 x 9 = 189	48 x 2 = 96
27 x 8 = 216	43 x 3 = 129	24 x 3 = 72
46 x 6 = 276	71 x 6 = 426	19 x 8 = 152
35 x 4 = 140	52 x 3 = 156	37 x 2 = 74
69 x 2 = 138	31 x 7 = 217	14 x 6 = 84
74 x 3 = 222	93 x 3 = 279	46 x 2 = 92
12 x 3 = 36	90 x 7 = 630	32 x 2 = 64
60 x 8 = 480	21 x 4 = 84	30 x 2 = 60
33 x 3 = 99	20 x 4 = 80	

LARGER MULTIPLIERS TIMES ONE-DIGIT MULTIPLICANDS

Multiplication of three-four-five-digit (etc.) multipliers times one-digit multiplicands follows the same basic pattern as is used above. We simply multiply the multiplicand by every digit in the multiplier, starting with the far left digit and proceeding consecutively to the right. Try these problems.

642 x 8 = 5136	482 x 3 = 1446	285 x 4 = 1140
232 x 6 = 1392	973 x 2 = 1946	378 x 3 = 1134
324 x 7 = 2268	591 x 8 = 4728	164 x 7 = 1148
667 x 2 = 1334	642 x 4 = 2568	143 x 3 = 429
365 x 4 = 1460	327 x 4 = 1308	214 x 2 = 428
839 x 7 = 5873	836 x 3 = 2508	270 x 8 = 2160
946 x 8 = 7568	417 x 9 = 3753	460 x 5 = 2300
473 x 6 = 2838	546 x 2 = 1092	308 x 2 = 616
902 x 6 = 5412	603 x 3 = 1809	

7835 x 6 = ~~44610~~ 47010	6543 x 8 = 52344	4782 x 9 = 43038
5378 x 7 = 37646	6943 x 3 = 20829	4827 x 4 = 19308
5379 x 2 = 10758	1732 x 8 = 13856	1940 x 7 = 13580
8700 x 5 = 43500	9000 x 6 = 54000	8305 x 3 = 24915
4072 x 4 = 16288		

67894 x 8 = 543152	53762 x 6 = 322572
42863 x 3 = 128589	27041 x 4 = 108164
54705 x 7 = 452935	30006 x 2 = 60012
40000 x 8 = 320000	19020 x 6 = 114120

TWO-DIGIT MULTIPLIER TIMES TWO-DIGIT MULTIPLICAND

Now we are ready to study multiplication involving two-digit multipliers and two-digit multiplicands.

We must be certain to continue setting the multiplicand correctly.

Use proper finger motions.

Use position concept correctly.

When doing this type of multiplication, follow this procedure.

43 x 76

1. Multiply the <u>first</u> digit of the multiplier (4) times the last digit of the multiplicand (6).
2. Set the product (24) in the first position (columns 1 and 2) to the right of the multiplicand.
3. Multiply the second digit of the multiplier (3) times the <u>last</u> digit of the multiplicand (6).
4. Set the product (18) in the second position (columns 2 and 3) to the right of the multiplicand.
5. Clear the last digit of the multiplicand (6) since it has now been multiplied by every number in the multiplier.
6. Now multiply the first digit of the multiplier (4) times the now <u>last</u> digit of the multiplicand. (7).
7. Set the product (28) in the now first position to the right of the remaining multiplicand. (If your calculation thus far are correct, you will set the 2 of 28 in the empty column immediately to the right of the 7. You will add the eight to the next column by clearing 2 and setting one left.

To the right of the seven you should now have 3058.

8. Next multiply the second digit of the multiplier (3) times the now last digit of the multiplicand. (7).
9. Set the product (21) in the second position to the right of the remaining multiplicand.
10. Clear the multiplicand (7) since it has been multiplied by all digits of the multiplier.
11. The final product, 3268, is now shown on the abacus.

Now try the next problem, using the same procedure. The instructions here are much more concise. If you find you need more detailed help, turn again to the problem just completed.

83 x 94

1. First (8) times last (4).
   Product (32) first position.
2. Second (3) times last (4).
   Product (12) second position.
3. Clear last of multiplicand (4).
4. First (8) times now last (9).
   Product (72) now first position.
5. Second (3) times now last (9).
   Product (27) now second position.
6. Clear now last of multiplicand (9).
7. Answer - 7802

To gain speed and accuracy in the multiplication of two-digit

multipliers times two-digit multiplicands, work the following problems.

79 x 25 = 1975	37 x 58 = 2146	47 x 39 = 1833
82 x 96 = 7872	58 x 37 = 2146	67 x 24 = 1608
26 x 79 = 2054	45 x 73 = 3285	
26 x 93 = 2418	67 x 81 = 2427	48 x 72 = 3456
35 x 53 = 1855		
71 x 89 = 6319	62 x 74 = 4588	53 x 92 = 4876
91 x 67 = 6097		
37 x 28 = 1036	46 x 24 = 1104	19 x 86 = 1634
14 x 57 = 798		
23 x 31 = 713	34 x 21 = 714	
90 x 71 = 6390	84 x 50 = 4200	30 x 21 = 630
67 x 80 = 5360	40 x 35 = 1400	70 x 86 = 6020
42 x 60 = 2520	31 x 20 = 620	

LARGER MULTIPLIERS TIMES LARGER MULTIPLICANDS

Regardless of how many digits in either the multiplier or multiplicand, the multiplication procedure is basically the same. You always begin with the <u>first</u> digit of the multiplier times the <u>last</u> digit of the multiplicand. You continue by multiplying that last digit of the multiplicand times each consecutive digit of the multipliers through the last one. Then clear that last digit, and repeat the process starting again with the first digit of the multiplier. Continue until all digits of the multiplicand have been cleared. Also, you must remember that the product of the first multiplication of each multiplicand goes into the present first position, the second product into

the second position, and so on.

Next, let's consider briefly the multiplication procedure for three-digit multipliers and two-digit multiplicands.

374 x 59

1. First (3) times last (9) - product (27) first position.
2. Second (7) times last (9) - product (63) second position.
3. Third (4) times last (9) - product (36) third position.
4. Clear Last of multiplicand (9).
5. First (3) times now last (5) - product (15) now first position.
6. Second (7) times now last (5) - product (35) now second position.
7. Third (4) times now last (5) - product (20) now third position.
8. Clear now last of multiplicand.
9. Answer - 22066

Do these problems for practice. Some of them have three-digit multipliers, some have four-digit, some five. The procedure for all of them is nearly the same.

476 x 38 = 18088	864 x 72 = 62208	400 x 40 = 16000
593 x 84 = 49812	971 x 86 = 83506	270 x 10 = 2700
725 x 95 = 68875	473 x 61 = 28853	493 x 20 = 9860
368 x 27 = 9956	640 x 87 = 55680	924 x 36 = 33264
702 x 63 = 44226		

3674 x 98 = 360052	6429 x 73 = 469317	4060 x 87 = 353220
7486 x 39 = 291954	7581 x 64 = 485184	6024 x 13 = 78312
5738 x 64 = 367232	8963 x 21 = 188223	3000 x 30 = 90000

2657 x 38 = 100966	4870 x 36 = 175320	7000 x 60 = 420000
4389 x 26 = 114114	7306 x 55 = 387218	
67849 x 35 = 2374715	72031 x 34 = 2449054	

If we increase the number of digits in the multiplicand, the basic multiplication procedure remains the same. We begin by multiplying the first digit of the multiplier times the last digit of the multiplicand. Then we continue to multiply that last digit of the multiplicand by every consecutive digit of the multiplier. Then we clear that last digit and repeat the whole process for each new last digit of the multiplicand.

Do these problems for practice.

47 x 396 = 18612	83 x 396 = 32868	23 x 401 = 9223
63 x 784 = 49392	75 x 184 = 13800	84 x 600 = 50400
86 x 437 = 37582	32 x 235 = 7520	34 x 402 = 13668
29 x 863 = 25027	19 x 308 = 5852	70 x 500 = 35000
35 x 927 = 32445	20 x 974 = 19480	
49 x 6783 = 332367	37 x 2639 = 97643	263 x 4789 = 1259507
73 x 4698 = 342954	91 x 7268 = 661388	5496 x 746 = 4098216
58 x 7436 = 431288	32 x 4321 = 138272	4691 x 875 = 4068315
842 x 679 = 571718	67 x 8404 = 563068	321 x 1234 = 396114
456 x 793 = 361608	381 x 243 = 92583	6070 x 348 = 2112360
374 x 985 = 368390	436 x 327 = 142572	809 x 4902 = 3965718
653 x 478 = 312134	760 x 894 = 679440	600 x 8000 = 4800000

ADDITIONAL PRACTICE

For additional practice in multiplication work these problems with interesting products. You will be able to set only the multiplicand on the abacus since there is not room enough for both it and the multiplier. The multipliers here are short, however, and you should be able to remember them easily.

18 x 123,456,789 = 2,222,222,202
27 x 123,456,789 = 3,333,333,303
36 x 123,456,788 = 4,444,444,404
45 x 123,456,789 = 5,555,555,505
54 x 123,456,789 = 6,666,666,606
63 x 123,456,788 = 7,777,777,707
72 x 123,456,789 = 8,888,888,808
81 x 123,456,789 = 9,999,999,909

9 x 9 = 81
81 x 9 = 729
729 x 9 = 6561
6561 x 9 = 59049
59049 x 9 = 531441
531441 x 9 = 4782969

8 x 8 = 64
64 x 8 = 512
512 x 8 = 4096
4096 x 8 = 32768
32768 x 8 = 262144
262144 x 8 = 2097152

7 x 9 = 49
49 x 7 = 343
343 x 7 = 2401
2401 x 7 = 16807
16807 x 7 = 117649
117649 x 7 = 823543

6 x 6 = 36
36 x 6 = 216
216 x 6 = 1296
1296 x 6 = 7776
7776 x 6 = 46656
46656 x 6 = 279936

5 x 5 = 25
25 x 5 = 125
125 x 5 = 625

4 x 4 = 16
16 x 4 = 64
64 x 4 = 256

3 x 3 = 9
9 x 3 = 27
27 x 3 = 81

625 x 5 = 3125	256 x 4 = 1024	81 x 3 = 243
3125 x 5 = 15625	1024 x 4 = 4096	243 x 3 = 729
15625 x 5 = 78125	4096 x 4 = 16384	729 x 3 = 2187

2 x 2 = 4

4 x 2 = 8

8 x 2 = 16

16 x 2 = 32

32 x 2 = 64

64 x 2 = 128

MULTIPLICATION OF DECIMALS

When multiplying decimals, simply proceed as if the decimal points were not present. Once you obtain the final product count the number of decimal places in the multiplier and multiplicand combined; then count off that many decimal places in the product.

Thus,

435 x 79 = 34365 is worked the same as 43.5 x 79 = 3436.5 or 43.5 x 7.9 = 343.65 or 4.35 x .79 = 3.4365; but the number of decimal places in the products differ.

When either the multiplier or multiplicand is less than one and has one or more zeroes as place holders between the decimal and the first value digit (.04, .007, .075, .0006, .000675), then, in counting digits to determine the placement of the multiplicand, count only the number of value digits, and then add one as usual. Thus, .07 x .08 would have the multiplicand set on the <u>third</u> column from the right

(1 value digit in multiplier + 1 value digit in multiplicand +1 = 3).

Do the following problems for practice in the multiplication of decimals.

4.2 x 6.87 = 28.854	.65 x 4.02 = 2.6130
$.34 x $6.75 = $2.2950	36 x 8.07 = 290.52
7.4 x .348 = 2.5752	5.9 x .007 = .0413
$8.95 x $6.07 = $54.3265	8.15 x .057 = .46455
.78 x .3 = .234	7.01 x .08 = .1608
7.06 x 3.001 = 21.18706	.006 x 3.7 = .0222
$1.23 x $3.21 = $3.0483	.09 x 6.75 = .6075
5.9 x 6.783 = 40.0197	.007 x .016 = .000112
66.2 x 38.4 = 2542.08	.003 x .0068 = .0000204

SUBTRACTION

DIRECT SUBTRACTION

Subtraction, like addition, is done either directly or with the use of secrets. In subtraction, the number from which another number is subtracted is the minuend. The number that is subtracted from that minuend is the subtrahend. The difference or answer is the remainder.

As in addition, subtraction on the abacus is done from left to right. Finger usage remains the same except for one change which will be demonstrated in example #2 below.

Example #1 of Direct Subtraction.

873 - 321 = 552

1. Set 873 on the last three columns on the right side of the abacus.
2. On the hundreds column, with the right forefinger, clear 3 earth counters.
3. On the tens column, with the right forefinger, clear 2 earth counters.
4. On the units column, with the right forefinger, clear 1 earth counter.
5. The answer, 552, is now shown on the abacus.

Example #2

798 - 678 = 120

1. Set 798 on the last three columns at the right of the abacus.
2. On the hundreds column, with one movement, clear the heaven counter with the right forefinger and clear one earth counter with the right thumb. (Notice this one change in finger usage.)
3. On the tens column, with one movement, clear the heaven counter with the right forefinger and clear two earth counters with the thumb.

4. On the units column, with one movement, clear the heaven counter with the right forefinger and clear three earth counters with the thumb.

5. The answer 120 is shown on the abacus.

Work the following problems as practice in simple subtraction. Always use the proper fingers, and always work from left to right.

94	78	63	87	96	68	794	597	389	7964	8642	9726
-32	-21	-52	-26	-46	-18	-632	- 86	-286	-6953	-7641	-6705
62	57	11	61	50	50	162	511	103	1011	1001	3021

5707	47865	69508
-4702	-46860	-68007
1005	1005	1501

SECRETS

Now we will consider subtraction using secrets as we did with addition. Once again, it is important that you memorize the secrets until they become automatic. The seventeen secrets used in subtraction are listed below. Study them carefully.

To Subtract	Secret
1	Clear 1 left, Set 9
1	Set 4, Clear 5
2	Clear 1 left, Set 8
2	Set 3, Clear 5
3	Clear 1 left, set 7
3	Set 2, Clear 5
4	Clear 1 left, Set 6

To Subtract	Secret
4	Set 1, Clear 5
5	Clear 1 left, Set 5
6	Clear 1 left, Set 4
6	Clear 1 left, Set 5, Clear 1
7	Clear 1 left, Set 3
7	Clear 1 left, Set 5, Clear 2
8	Clear 1 left, Set 2
8	Clear 1 left, Set 5, Clear 3
9	Clear 1 left, Set 1
9	Clear 1 left, Set 5,Clear 4

SECRETS FOR SUBTRACTING ONE

First we will study the secrets for subtracting 1.

Clear 1 left, Set 9

Set 4, Clear 5

For practice, set the number 999 and repeatedly subtract 1 until you reach 0. Proceed in this manner.

1. Set 999 on the last three columns at the right of the abacus.
2. On the units column, clear one earth counter with the right forefinger.
3. On the units column, clear one earth counter with the right forefinger.
4. On the units column, clear one earth counter with the right forefinger.
5. On the units column, clear one earth counter with the right

forefinger.

6. On the units column, set four earth counters with the thumb and clear one heaven counter with the forefinger. (Set 4, Clear 5).

7. On the units column, clear one earth counter with the forefinger.

8. On the units column, clear one earth counter with the forefinger.

9. On the units column, clear one earth counter with the forefinger.

10. On the units column, clear one earth counter with the forefinger.

11. On the tens column, clear one earth counter with the right forefinger. Then with one movement, on the units column set the heaven counter with the forefinger and set 4 earth counters with the thumb. (Clear 1 left, Set 9).

Contine on in this manner until you master the secrets for subtracting one. Then do these problems.

34	78	55	20	50	95	80	759	705	850
-11	-11	-11	-11	-11	-11	-11	-111	-111	-111
23	67	44	9	39	84	69	648	594	739

Finally, set the number	9,987,654,321
Then subtract	- 111,111,111
	9,876,543,210
Again subtract	- 111,111,111
	9,765,432,099
Again subtract	- 111,111,111
	9,654 320,988
Again subtract	- 111,111,111
	9,543,209,877

Again subtract	9,543,209,877
	- 111,111,111
	9,432,098,766
Again subtract	- 111,111,111
	9,320,987,655
Again subtract	- 111,111,111
	9,209,876,544
Again subtract	- 111,111,111
	9,098,765,433
Again subtract	- 111,111,111
	8,987,654,322

SECRETS FOR SUBTRACTING TWO

Now we will consider the secrets for subtracting 2.

Clear 1 left, Set 8

Set 3, Clear 5

1. Set 999
2. Clear 2
3. Clear 2
4. Set 3, Clear 5
5. Clear 2
6. Clear 1 left, Set 8
7. Clear 2
8. Clear 2
9. Set 3, Clear 5
10. Clear 2
11. Clear 1 left, Set 8

Etc.

For practice, do these problems.

34	97	65	56	75	96	686	567	945	865	91	71	40
-22	-22	-22	-22	-22	-22	-222	-222	-222	-222	-22	-22	-22
12	75	43	34	53	74	464	345	723	643	69	49	18

31	910	701	815	6081	5316	54610	68501	40651	71605
-22	-222	-222	-222	-2222	-2222	-22222	-22222	-22222	-22222
9	688	479	593	3859	3094	32388	46279	18429	49383

91670
-22222
69448

Finally, set the number 9,987,654,321. From it subtract 222,222,222 nine consecutive times and obtain these results:

9,765,432,099
9,543,209,877
9,320,987,655
9,098,765,433
8,876,543,211
8,654,320,989
8,432,098,767
8,209,876,545
7,987,654,323

SECRETS FOR SUBTRACTING THREE

Next we will practice the secrets for subtracting 3.

Clear 1 left, Set 7

Set 2, Clear 5

1. Set 999
2. Clear 3
3. Set 2, Clear 5
4. Clear 3
5. Clear 1 left, Set 7
6. Set 2, Clear 5

8. Clear 3

9. Clear 1 left, Set 7

10. Clear 3

11. Set 2, Clear 5

12. Clear 1 left, Set 7

Etc.

For more practice in subtracting 3, do these problems.

34	98	86	59	65	76	57	576	475	635	547	42	90
-33	-33	-33	-33	-33	-33	-33	-333	-333	-333	-333	-33	-33
1	65	53	26	32	43	24	243	142	302	214	9	57

81	91	71	62	92	712	361	5261	62517	85760	91526
-33	-33	-33	-33	-33	-333	-333	-3333	-33333	-12321	-32332
48	58	38	29	59	379	28	1928	29184	73439	59194

74125	65107
-32232	-21333
41893	43774

Finally, set the number 9,987,654,321. Then subtract 333,333,333 from the number nine consecutive times.

9,654,320,988
9,320,987,655
8,987,654,322
8,654,320,989
8,320,987,656
7,987,654,323
7,654,320,990
7,320,987,657
6,987,654,324

For added practice in subtracting 1, 2, and 3, set 9,999,999,999,999. Then subtract 123,123,123,123 nine consecutive times obtaining these remainders.

9,876,876,876,876
9,753,753,753,753
9,630,630,630,630
9,507,507,507,507
9,384,384,384,384
9,261,261,261,261
9,138,138,138,138
9,015,015,015,015
8,891,891,891,892

SECRETS FOR SUBTRACTING FOUR

Now study the secrets for subtracting 4.

Clear 1 left, Set 6

Set 1, Clear 5

1. Set 999
2. Clear 4
3. Set 1, Clear 5
4. Clear 1 left, Set 6
5. Set 1, Clear 5
6. Clear 1 left, Set 6
7. Clear 4
8. Set 1, Clear 5
9. Clear 1 left, Set 6

Etc.

Practice these problems.

94	49	64	86	67	98	87	678	597	685	82	71	93
-44	-44	-44	-44	-44	-44	-44	-444	-444	-444	-44	-44	-44
50	5	20	42	23	54	43	234	153	241	38	27	49

125	612	327	740	631	761	821	813	413	671	8631
- 44	-444	-44	-444	-444	-444	-444	-444	- 44	-444	-4444
81	168	283	296	187	317	377	369	369	227	4187

7538	57138	72032
-4444	-44444	-44444
3094	12694	27588

Finally set the number 9,987,654,321. Then subtract 444,444,444 nine consecutive times obtaining these remainders.

9,543,209,877
9,098,765,433
8,654,320,989
8,209,876,545
7,765,432,101
7,320,982,657
6,876,543,213
6,432,098,769
5,987,654,325

SECRET FOR SUBTRACTING FIVE

Next consider the secret for subtracting 5.

Clear 1 left, Set 5

1. Set 999
2. Clear 5
3. Clear 1 left, Set 5
4. Clear 5
5. Clear 1 left, Set 5
6. Clear 5
7. Clear 1 left, Set 5

Etc.

Practice these problems.

55	95	58	73	84	61	92	382	724	612	213	671	703
-55	-55	-55	-55	-55	-55	-55	- 55	-555	-555	- 55	-555	-555
00	40	3	18	29	6	37	327	169	57	158	116	148

602	800	1207	7203	3010
-555	-555	- 555	- 555	- 555
47	245	652	6648	2455

Finally, set the number 9,987,654,321 and subtract 555,555,555 nine consecutive times obtaining these remainders.

9,432,098,766
8,876,543,211
8,320,987,656
7,765,432,101
7,209,876,546
6,654,320,991
6,098,765,436
5,543,209,881
4,987,654,326
4,432,098,771

SECRETS FOR SUBTRACTING SIX

Next study the secrets for subtracting 6.

Clear 1 left, Set 4

Clear 1 left, Set 5, Clear 1

1. Set 999

2. Clear 6, on the units column, with one movement, clear the heaven counter with the right forefinger and clear one earth counter with the thumb.

3. Clear 1 left, Set 5, Clear 1. On the tens column clear one earth counter with the right forefinger. Then on the units column, set one heaven counter with the forefinger and continue on to clear 1 earth counter on the same column, with the same finger.

4. Clear 6

5. Clear 1 left, Set 5, Clear 1

6. Clear 1 left, Set 4. On the tens column, clear 1 earth counter with the forefinger. At the same time on the

units column set 4 earth counters with the thumb.

7. Clear 6

8. Clear 1 left, Set 5, Clear 1

Etc.

Do these problems to practice subtracting 6.

98	76	70	85	95	805	350	495	73	82	91	842	732
-66	-66	-66	-66	-66	- 66	- 66	- 66	-66	-66	-66	- 66	- 66
32	10	4	19	29	739	284	429	7	16	25	776	666

625	803	410	307	954	125	67152	48105	84015	70401
- 66	-666	- 66	- 66	-666	- 66	-66666	- 6666	-66666	-66666
559	137	344	241	288	59	486	41439	17349	3735

95402
-66666
28736

Finally, set the number 9,987,654,321 and subtract from it the number 666,666,666 nine consecutive times obtaining these remainders.

9,320,987,655
8,654,320,989
7,987,654,323
7,320,987,637
6,654,320,991
5,987,654,325
5,320,987,659
4,654,320,993
3,987,654,327

For practice subtracting the numbers 1 through 6, practice this problem until the use of the secrets become automatic.

Set the number 9,999,999,999,999 and from it subtract the number 123,456,123,456 nine consecutive times to obtain these remainders.

9,876,543,876,543
9,753,087,753,087
9,629,631,629,631
9,506,175,506,175
9,382,719,382,719
9,259,263,259,263
9,135,807,135,807
9,012,351,012,351
8,888,894,888,895

SECRETS FOR SUBTRACTING SEVEN

Next study the secrets for subtracting 7.

Clear 1 left, Set 3

Clear 1 left, Set 5, Clear 2

1. Set 999
2. Clear 7
3. Clear 1 left, Set 5, Clear 2
4. Clear 1 left, Set 3
5. Clear 7
6. Clear 1 left, Set 3
7. Clear 1 left, Set 5, Clear 2
8. Clear 1

Etc.

Do these problems to practice subtracting 7.

78	98	81	90	281	461	315	216	501	710	84	93	82
-77	-77	-77	-77	-77	-77	-77	-77	-77	-77	-77	-77	-77
1	21	4	13	204	384	238	139	424	633	7	16	5

128	247	634	542	332	824	174	412	603	281	121	440
-77	-77	-77	-77	-77	-77	-77	-77	-77	-77	-77	-77
51	170	557	465	255	747	97	335	526	204	44	363

302	14731	56012
-77	-7777	-7777
225	6954	48235

Finally, set the number 9,987,654,321 and subtract from it the number 777,777,777 nine consecutive times obtaining these remainders:

9,209,876,544
8,432,098,767
7,654,320,990
6,876,543,213
6,098,765,436
5,320,987,659
4,543,209,882
3,765,432,105
2,987,654,328

SECRETS FOR SUBTRACTING EIGHT

Now study the secrets for subtracting 8.

Clear 1 left, Set 2

Clear 1 left, Set 5, Clear 3

1. Set 999
2. Clear 8
3. Clear 1 left, Set 2,
4. Clear 1 left, Set 5, Clear 3
5. Clear 1 left, Set 2
6. Clear 1 left, Set 2
7. Clear 8
8. Clear 1 left, Set 2
9. Clear 1 left, Set 5, Clear 3

Etc.

Do these problems in subtracting 8.

98	89	91	125	161	450	371	625	807	265	93	94
-88	-88	-88	- 88	- 88	- 88	- 88	- 88	- 88	- 88	-88	-88
10	1	3	37	73	362	283	537	719	177	5	6

143	243	633	834	443	544	734	241	623	236	147
- 88	- 88	- 88	- 88	- 88	- 88	- 88	- 88	- 88	- 88	- 88
55	155	545	746	355	456	646	153	535	148	59

24374	65463	53147
- 8888	-8888	- 8888
15486	56575	44259

Finally, set the number 9,987,654,321 and subtract from it the number 888,888,888 nine consecutive times obtaining these remainders.

9,098,765,433
8,209,876,545
7,320,987,657
6,432,098,769
5,543,209,881
4,654,320,993
3,765,432,105
2,876,543,217
1,987,654,329

SECRETS FOR SUBTRACTING NINE

Next look at the secrets for subtracting 9.

Clear 1 left, Set 1

Clear 1 left, Set 5, Clear 4

1. Set 999
2. Clear 9
3. Clear 1 left, Set 1
4. Clear 1 left, Set 1
5. Clear 1 left, Set 1
6. Clear 1 left, Set 1
7. Clear 1 left, Set 5, Clear 4

8. Clear 1 left, Set 1

9. Clear 1 left, Set 1

10. Clear 1 left, Set 1

11. Clear 1 left, Set 1

12. Clear 9

Etc.

For practice in subtracting 9, find these remainders:

199	289	487	363	572	781	405	370	144	344	124	604
- 99	- 99	- 99	- 99	- 99	- 99	- 99	- 99	- 99	- 99	- 99	- 99

784	546	442	534	347	6943	2684	3041	6740
- 99	- 99	- 99	- 99	- 99	- 999	- 999	- 999	- 999

Finally, set the number 9,987,654,321 and from it subtract the number 999,999,999 nine consecutive times obtaining these remainders.

8,987,654,322
7,987,654,323
6,987,654,324
5,987,654,325
4,987,654,326
3,987,654,327
2,987,654,328
1,987,654,329
987,654,330

ADDITIONAL PRACTICE

For extra practice in subtracting the numbers 1 through 9 set the number 9,999,999,909, and subtract from it the number 123,456,789 several consecutive times obtaining the following remainders:

9,999,999,909	6,666,666,606	3,333,333,303
9,876,543,120	6,543,209,817	3,209,876,514
9,753,086,331	6,419,753,028	3,086,419,725
9,629,629,542	6,296,296,239	2,962,962,936
9,506,172,753	6,172,839,450	2,839,506,147

9,382,715,964	6,049,382,661	2,716,049,358
9,259,259,175	5,925,925,872	2,592,592,569
9,135,802,386	5,802,469,083	2,469,135,780
9,012,345,597	5,679,012,294	2,345,678,991
8,888,888,808	5,555,555,505	2,222,222,202
8,765,432,019	5,432,098,716	2,098,765,413
8,641,975,230	5,308,641,927	1,975,308,624
8,518,518,441	5,185,185,138	1,851,851,835
8,395,061,652	5,061,728,349	1,728,395,046
8,271,604,863	4,938,271,560	1,604,938,257
8,148,148,074	4,814,814,771	1,481,481,468
8,024,691,285	4,691,357,982	1,358,024,679
7,901,234,496	4,567,901,193	1,234,567,890
7,777,777,707	4,444,444,404	1,111,111,101
7,654,320,918	4,320,987,615	987,654,312
7,530,864,129	4,197,530,826	864,197,523
7,407,407,340	4,074,074,037	740,740,734
7,283,950,551	3,950,617,248	617,283,945
7,160,493,762	3,827,160,459	493,827,156
7,037,036,973	3,703,703,670	370,370,367
6,913,580,184	3,580,246,881	246,913,578
6,790,123,395	3,456,790,092	123,456,789

After each intense practice using the secrets for subtraction, you may find that a review of the secrets for addition is necessary. Return to the section on addition and study the secrets carefully. Then return to this section and practice some of the problems again - first subtracting, and then adding again. For instance, 27 - 9 = 18 then 18 + 9 = 27. Next start with 9,999,999,909 as you did earlier. This time, subtract 123,456,789 from it nine consecutive times to get 8,888,888,808. Then add 123,456,789 to that number nine times to get 9,999,999,909. Practice in this manner until you are confident in your use of the secrets for both addition and subtraction.

One other helpful exercise is to

add	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45
then to subtract	45 - 9 - 9 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 0
	45 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 0

1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 46
46 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 1
1 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 46
46 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 1

2 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 47
47 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 2
2 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 47
47 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 2

3 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 48
48 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 3
3 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 48
48 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 3

4 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 49
49 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 4
4 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 - 49
49 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 4

5 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 50
50 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 = 5
5 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 50
50 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 = 5

SUBTRACTION WITH DECIMALS

The same secrets are used in the subtraction of decimals as are used in the subtraction of whole numbers. As in addition, however, there is a specific procedure for determining the proper placement of decimal numbers on the abacus. This procedure for placing the numbers is the same as used in addition, and therefore, will not be dealt with in great detail here.

1. First find the number having the most decimal places.
2. If most decimal places is 3 or less, use <u>first</u> unit mark as decimal point.
3. If most decimal places is 4 to 6, use <u>second</u> unit mark as decimal point.

4. If most decimal places are 7 to 9, use <u>third</u> unit mark as decimal point.

Remember to set both the minuend and the subtrahend so that their decimal points fall in the proper position. Then merely use the secrets for subtraction which you have already mastered. Study the example below and work the following problems.

Example # 1

84.7259 - 6.409 =

Largest number of decimal places is 4, second unit mark is the decimal point.

84.7259	Set so that 4 is on column immediately to left 2nd unit mark.
- 6.408	Set so that 6 is on column immediately to left of 2nd unit mark.
78.3179	The remainder 8 is on column immediately to left of 2nd unit mark.

6.74	53.8	61.8	478.94	132.94	24.3	5.6431	63.87941
-1.92	- 9.6	-24.76	- 34.549	- 68.743	-18.7	-2.097	- 8.4379
4.82	44.23	37.04	444.391	64.197	5.6	3.5561	55.44151

94.6	4.009	1.0007458	24.00068432	12.65
-3.0075	-2.0001	.00643	- 6.5	-8.0439
91.5925	2.0089	.9943158	17.50068432	4.6061

WHEN SUBTRAHEND IS LARGER THAN MINUEND

When the subtrahend is larger than the minuend, as for example, in the problem 47 - 94 = , the problem can still be worked on the abacus. Use this procedure:

1. Set the subtrahend first 94.

2. Subtract from it, in the usual manner, the minuend 47.

3. Place a minus sign in front of the remainder -41. It is the correct answer.

Do these problems for practice:

14	35	71	24	42	12	643	247	891	687	421	4765	2714
-65	-94	-96	-61	-75	-34	-851	-596	-964	-860	-568	-6843	-4623
-51	-59	-25	-37	-33	-22	-208	-349	- 73	-173	-147	-2078	-1909

9642	6432	4761
-9871	-8465	-9432
- 229	-2033	-4671

DIVISION

Before we can discuss the procedures used in dividing on the abacus, it is essential that we understand the names of the various parts of the problem, their functions, and where they are placed on the abacus.

The divisor is the number that we divide into another number. It is set as far left on the abacus as possible. The dividend is the number into which we are dividing. It is set as far right on the abacus as possible. The quotient is the answer, and it is set on the columns to the left of the dividend. Exactly how it is positioned is a very important concept and must be thoroughly understood before proceeding.

When dividing with a one-digit divisor, if the divisor can be divided into the first digit of the dividend (6 ÷ 3, 34 ÷ 2, 746 ÷ 4) then the quotient is placed on the second column to the left of the dividend. If the one-digit divisor can not be divided into the first digit of the dividend (18 ÷ 6, 247 ÷ 7, 438 ÷ 6) then the quotient is placed on the column immediately to the left of the dividend.

When dividing with a two-digit divisor, if the divisor can be divided into the first two digits of the dividend (36 ÷ 18, 516 ÷ 24), then the quotient is placed on the second column to the left of the dividend. If the two-digit divisor cannot be divided into the first two digits of the dividend (146 ÷ 18, 428 ÷ 50), then the quotient is set on the column immediately to the left of the dividend.

When three-digit divisor is used, if it can be divided into the first three digits of the dividend, the quotient is set on the second column to the left; if it cannot, it is set immediately to the left.

The concept is the same for larger divisors.

ONE-DIGIT DIVISORS

First, we will consider division problems having only one digit in the divisor, and one, two, three, or more digits in the dividend. Follow the following procedure carefully to work the first example. Remember to set and clear with the proper fingers.

$8 \div 4 =$

1. Set the <u>divisor</u> (4) on the column farthest left.
2. Set the <u>dividend</u> (8) on the column farthest right.
3. Divide the dividend (8) by the divisor (4). Set the quotient (2) on the second column to the left of the dividend. (8) (Remember when a one-digit divisor <u>can</u> be divided into the first digit of the dividend, the quotient is set on the second column to the left of the dividend.)
4. Multiply the quotient (2) times the divisor (4). (Remember that in multiplication, one-digit products are set in the second column of the position.) Subtract product (8) from the dividend (8).
5. The final quotient (2) is now shown on the abacus. We know that the quotient is 2 and not 20, or 200, by determining

which column is the last one in the product. To determine this, we add the number of digits in the divisor (1) plus one more. In this case the sum is 2. We then count off two columns from the right edge of the abacus. The next column to the left is the last digit of the quotient.

Divide these problems. Concentrate on establishing as habit the simple procedure above. Doing so will help when you begin to divide larger numbers.

6 ÷ 3 = 2	2 ÷ 1 = 2	6 ÷ 6 = 1
9 ÷ 3 = 3	3 ÷ 3 = 1	5 ÷ 5 = 1
6 ÷ 2 = 3	8 ÷ 8 = 1	4 ÷ 4 = 1
4 ÷ 2 = 2	8 ÷ 2 = 4	3 ÷ 3 = 1

Now study carefully this example which still has a one-digit divisor but a two-digit dividend.

18 ÷ 6 =

1. Set the divisor (6) on the column farthest left.
2. Set the dividend (18) on the two columns farthest right.
3. Divide the dividend (18) by the divisor (6). Set the quotient (3) on the column immediately to the left of the dividend (18) (Remember, when a one-digit divisor can <u>not</u> be divided into the <u>first</u> digit of the dividend the resulting quotient is <u>set</u> in the column immediately to the left of the dividend.)
4. Multiply the quotient (3) times the divisor (6). Subtract the product (18) from the dividend (18) which is, of

course, in the first position to the right.

5. The final quotient (3) is now shown on the abacus. Again, one digit in the divisor plus one more = 2, and thus the last digit of the quotient is in the 3rd column from the right.

Practice these problems before studying the next example.

12 ÷ 3 = 4	48 ÷ 6 = 8	16 ÷ 4 = 4
21 ÷ 7 = 3	63 ÷ 7 = 9	72 ÷ 8 = 9
36 ÷ 4 = 9	81 ÷ 9 = 9	30 ÷ 6 = 5
56 ÷ 8 = 7	24 ÷ 6 = 4	20 ÷ 4 = 5
15 ÷ 5 = 3	40 ÷ 5 = 8	64 ÷ 8 = 8

The next example also has a one-digit divisor and a two-digit dividend, but now the divisor <u>can</u> be divided into the <u>first</u> digit of the dividend. Therefore two divisions must be made.

86 ÷ 2 =

1. Set the divisor (2) on the column farthest left.
2. Set the dividend (86) on the columns farthest right.
3. Divide the divisor (2) into the first digit of the dividend (8) and set the quotient (4) on the <u>second</u> column to the left of the dividend (one-digit divisor can be divided into first digit of dividend).
4. Multiply the quotient (4) times the divisor (2) and subtract the product (8) from the first digit of the dividend (8). (Notice the product is one-digit and is subtracted from the 2nd column of the first position.)

5. Divide the divisor (2) into the now first digit of the dividend (6). Set the quotient (3) in the second column to the left of the remaining dividend. (One-digit divisor <u>can</u> again be divided into now first digit of dividend.)
6. Multiply the last quotient (3) times the divisor (2) and subtract the product (6) from the first digit of the remaining dividend.
7. The answer (43) is now shown on the abacus.

Do these problems for practice. Remember to determine which column is the last in the product.

$48 \div 4 = 12$	$66 \div 6 = 11$	$93 \div 3 = 31$
$69 \div 3 = 23$	$39 \div 3 = 13$	$60 \div 3 = 20$
$55 \div 5 = 11$	$84 \div 4 = 21$	$80 \div 8 = 10$
$28 \div 2 = 14$	$77 \div 7 = 11$	$20 \div 2 = 10$

Now study carefully the following example - one minor difference is noted.

$72 \div 4 =$

1. Set the divisor (4) on the column farthest left.
2. Set the dividend (72) on the columns farthest right.
3. Divide the divisor (4) into the first digit of the dividend (7) Set the quotient (1) on the second column to the left.
4. Multiply the quotient (1) times the divisor (4); subtract the product (4) from the first position. (Notice that sub-

tracting the product (4) does not, in this case, clear the entire first digit of the dividend as it has in all previous cases. Notice also that the division procedure doesn't change significantly. The remainder, after subtracting the product, merely becomes the first digit of the remaining dividend, and the division process continues as before).

5. Divide the divisor (4) into the now first two digits of the dividend (32). Set the quotient (8) in the column immediately to the left.
6. Multiply the quotient (8) times the divisor (4), subtract product (32) from the first position.
7. The answer, 18, is now shown on the abacus.

Work these problems for practice.

65 ÷ 5 = 13	72 ÷ 6 = 12
98 ÷ 7 = 14	72 ÷ 4 = 18
84 ÷ 3 = 28	91 ÷ 7 = 13
96 ÷ 6 = 16	48 ÷ 3 = 16
96 ÷ 8 = 12	50 ÷ 2 = 25
96 ÷ 4 = 24	85 ÷ 5 = 17
70 ÷ 5 = 14	68 ÷ 4 = 17

Next we will consider problems with one-digit divisors and three-or-more-digit dividends. The principles of division learned so far remain important - the only real difference is that more divisions are

required to work these larger problems.

Study these examples carefully before working the next group of problems.

216 ÷ 3 =

1. Set the divisor (3) on the column farthest left.
2. Set the dividend (216) on the columns farthest right.
3. Divide the divisor (3) into the first two digits of the dividend (21) and set the quotient (7) on the column immediately to the left.
4. Multiply the quotient (7) times the divisor (3) and subtract the product (21) from the first position which is the first two digits of the dividend (21).
5. Divide the divisor (3) into the first digit of the remaining dividend (6) and set the quotient (2) in the second column to the left.
6. Multiply the last quotient (2) times the divisor (3) and subtract the product (6) from the now first position.
7. The answer (72) is now shown on the abacus.

864 ÷ 2

1. Set the divisor (2) on the column farthest left.
2. Set the dividend (864) on the column farthest right.
3. Divide the divisor (2) into the first digit of the dividend (8) and set the quotient (4) on the second column to the left.

4. Multiply the quotient (4) times the divisor (2); subtract the product (8) from the first position.
5. Divide the divisor (2) into the now first digit of the dividend (6); set the quotient (3) on the second column to the left.
6. Multiply that quotient (3) times the divisor (2); subtract the product (6) from the now first position.
7. Divide the divisor (2) into the now first digit of the dividend (4); set the quotient (2) on the second column to the left.
8. Multiply the quotient (2) times the divisor (2); subtract the product (4) from the first position.
9. The answer (432) is now shown on the abacus.

812 ÷ 4 =

1. Set the divisor (4) on the column farthest left.
2. Set the dividend (812) on the column farthest right.
3. Divide the divisor (4) into the first digit of the dividend (8). Set the quotient (2) on the second column to the left.
4. Multiply the quotient (2) times the divisor (4); subtract the product (8) from the first position.
5. Divide the divisor (4) into the remaining two digits of the dividend (12); set the quotient (3) on the column immediately to the left.

6. Multiply the quotient (3) times the divisor (4); subtract the product from the now first position.

7. The answer, 203, is now shown on the abacus.

196 ÷ 7 =

1. Set the divisor (7) on the column farthest left.

2. Set the dividend (196) on the columns farthest right.

3. Divide the divisor (7) into the first two digits of the dividend (19); set the quotient (2) on the column immediately to the left.

4. Multiply the quotient (2) times the divisor (7); subtract the product (14) from the first position. (Notice that this subtraction does not completely clear the first two digits of the dividend. The remainder of 5 becomes the first digit of the dividend (56) and the division process continues.)

5. Divide the divisor (7) into the dividend (56); set the quotient (8) on the column immediately to the left.

6. Multiply the quotient (8) times the divisor (7); subtract the product (56) from the first position.

7. The answer, 28, is now shown on the abacus.

Once you understand the procedures described in the examples above, work the following problems for practice. Remember to determine which column is the last one in the product.

693 ÷ 3 = 231	2864 ÷ 2 = 1432	3876 ÷ 4 = 969
186 ÷ 6 = 31	6186 ÷ 6 = 1031	3416 ÷ 7 = 488
318 ÷ 3 = 106	3045 ÷ 5 = 609	8386 ÷ 7 = 1198
287 ÷ 7 = 41	8872 ÷ 8 = 1109	56826 ÷ 3 = 18942
540 ÷ 5 = 180	4974 ÷ 6 = 829	70216 ÷ 8 = 8777
540 ÷ 9 = 60	5284 ÷ 4 = 1321	60345 ÷ 5 = 12069
328 ÷ 4 = 82	1269 ÷ 9 = 141	836154 ÷ 9 = 92906
840 ÷ 8 = 105	8865 ÷ 3 = 2955	278128 ÷ 4 = 69532

700164 ÷ 6 = 116694	2791268 ÷ 4 = 697817
238200 ÷ 3 = 79400	4104000 ÷ 8 = 513000
747000 ÷ 9 = 83000	8301564 ÷ 6 = 1383594
700140 ÷ 7 = 100020	44295065 ÷ 5 = 8859013

REMAINDERS

Thus far in our discussion of division, all problems have divided evenly, and we have had no remainders. Problems with remainders are worked in exactly the same way as those we have already discussed. When all divisions have been made, we, as usual, determine which column is the last one in the quotient - any numbers to the right of that are the remainder.

Study the following example; then work the problems below.

435 ÷ 6 =

1. Set the divisor (6) on the column farthest left.
2. Set the dividend (435) on the column farthest right.

3. Divide the divisor (6) into the first two digits of the dividend (43). Set the quotient (7) on the column immediately to the left.

4. Multiply the quotient (7) times the divisor (6); subtract the product (42) from the first position.

5. Divide the divisor (6) into the now first two digits of the dividend (15); set the quotient (2) on the column immediately to the left.

6. Multiply the quotient (2) times the divisor (6); subtract the product (12) from the first position.

7. Determine which is the last column of the quotient - the answer 72 r3 is now shown on the abacus.

475 ÷ 6 = 79 r1	510 ÷ 8 = 63 r6	67947 ÷ 6 = 11324 r3
587 ÷ 4 = 146 r3	510 ÷ 4 = 127 r2	320017 ÷ 8 = 40002 r1
679 ÷ 8 = 84 r7	6109 ÷ 6 = 1018 r1	660045 ÷ 4 = 165011 r1
738 ÷ 5 = 147 r3	2106 ÷ 7 = 300 r6	54092 ÷ 3 = 18030 r2
687 ÷ 9 = 76 r3	2702 ÷ 3 = 900 r2	15003 ÷ 5 = 3000 r3
596 ÷ 7 = 85 r1	45004 ÷ 5 = 9000 r4	28074 ÷ 7 = 4010 r4

For additional practice in dividing with a one-digit divisor, work these sets of problems:

4782969 ÷ 9 = 531441	2097152 ÷ 8 = 262144	823543 ÷ 7 = 117649
531441 ÷ 9 = 59049	262144 ÷ 8 = 32768	117649 ÷ 7 = 16807
59049 ÷ 9 = 6561	32768 ÷ 8 = 4096	16807 ÷ 7 = 2401
6561 ÷ 9 = 729	4096 ÷ 8 = 512	2401 ÷ 7 = 343
729 ÷ 9 = 81	512 ÷ 8 = 64	343 ÷ 7 = 49
81 ÷ 9 = 9	64 ÷ 8 = 8	49 ÷ 7 = 7
9 ÷ 9 = 1	8 ÷ 8 = 1	7 ÷ 7 = 1

279936 ÷ 6 = 46656

46656 ÷ 6 = 7776

7776 ÷ 6 = 1296

1296 ÷ 6 = 216

216 ÷ 6 = 36

36   6 = 6

6 ÷ 6 = 1

78125 ÷ 5 = 15625

15625 ÷ 5 = 3125

3125 ÷ 5 = 625

625 ÷ 5 = 125

125 ÷ 5 = 25

25 ÷ 5 = 5

5 ÷ 5 = 1

16384 ÷ 4 = 4096

4096 ÷ 4 = 1024

1024 ÷ 4 = 256

256 ÷ 4 = 64

64 ÷ 4 = 16

16 ÷ 4 = 4

4 ÷ 4 = 1

2187 ÷ 3 = 729

729 ÷ 3 = 243

243 ÷ 3 = 81

81 ÷ 3 = 27

27 ÷ 3 = 9

9 ÷ 3 = 3

3 ÷ 3 = 1

128 ÷ 2 = 64

64 ÷ 2 = 32

32 ÷ 2 = 16

16 ÷ 2 = 8

8 ÷ 2 = 4

4 ÷ 2 = 2

2 ÷ 2 = 1

TWO-OR-MORE-DIGIT DIVISORS

Divison with two-or-more digits in the divisor follows much the same pattern as division with a one-digit divisor. A few steps are added to the procedure to deal with the added digits in the divisor, and these will be exemplified and explained below. Also, when dealing with two-or-more digit divisors, it is easy to mis-estimate when dividing, and ~~chose~~ choose a quotient which is either too large or too small. Ways for correcting such mis-estimates will be discussed.

Consider first a problem with a two-digit divisor and a two-digit dividend.

$98 \div 14 =$

1. Set the divisor (14) on the columns farthest left.
2. Set the dividend (98) on the column farthest right.
3. Divide the divisor (14) into the first two digits of the dividend (98) set the quotient (7) on the second column to the left of the dividend. (Recall that when a two-digit divisor <u>can</u> be divided into the first two digits of the dividend, the quotient is placed on the second column to the left.)
4. Multiply the quotient (7) times the first digit of the divisor (1) subtract the product (7) from the first position.
5. Multiply the quotient (7) times the second digit of the divisor (4); subtract the product (28) from the second position. (Notice that the position concept of multiplication is an integral part of division and must be used correctly.)
6. The answer, 7, is now shown on the abacus. (Finally, remember how to determine which is the last column of the quotient: number of digits in the divisor (2) plus one = three. Count in three columns from the right edge of the abacus - the next column to the left is the last digit of the quotient.

Work the following problems for practice. Some of them will have remainders: and, as before, after determining which is the last column of the quotient, any number to the right of that column will

represent a remainder.

32 ÷ 16 = 2	84 ÷ 42 = 2	99 ÷ 14 = 7 r1	63 ÷ 31 = 2 r1
75 ÷ 25 = 3	54 ÷ 27 = 2	98 ÷ 47 = 2 r4	54 ÷ 12 = 4 r6
84 ÷ 21 = 4	90 ÷ 18 = 5	59 ÷ 19 = 3 r2	73 ÷ 18 = 4 r1
96 ÷ 24 = 4	92 ÷ 23 = 4	68 ÷ 13 = 5 r3	87 ÷ 17 = 5 r2
30 ÷ 10 = 3	94 ÷ 47 = 2	46 ÷ 12 = 3 r10	39 ÷ 16 = 2 r7
45 ÷ 15 = 3	74 ÷ 37 = 2	78 ÷ 25 = 3 r3	76 ÷ 15 = 5 r1
39 ÷ 13 = 3	69 ÷ 23 = 3	71 ÷ 11 = 6 r5	24 ÷ 11 = 2 r2
70 ÷ 35 = 2	57 ÷ 19 = 3	38 ÷ 15 = 2 r8	48 ÷ 13 = 3 r9

CORRECTING MIS-ESTIMATES

As you may have already discovered in working the above problems, it is easy to mis-estimate and choose a quotient that is either too high or too low. It is, however, possible to correct such mis-estimates without starting the problem over. First, we will consider problems where the estimated quotient is too low. In our last example, we found the quotient of 98 ÷ 14 to be 7. Suppose, however, in 98 ÷ 14 = we had estimated the quotient to be 6.

1. Set the divisor (14) on the columns farthest left.
2. Set the dividend (98) on the columns farthest right.
3. Divide the divisor (14) into the first two digits of the dividend (98). Set the quotient (which we estimate at 6) on the second column to the left.
4. Multiply the quotient (6) times the first digit of the divisor (1); subtract the product (6) from the first position (leaving 38 in the dividend)

5. Multiply the quotient (6) times the second digit of the divisor (4); subtract the product (24) from the second position, (leaving 14 in the dividend).

At this point, we realize that our quotient estimate (6) was too low because the divisor (14) can be divided into the remainder (14). To correct this under-estimation:

6. Divide the divisor (14) into the remaining dividend (14). As usual set the quotient (1) on the second column to the left.
7. Multiply the quotient (1) times the first digit of the divisor (1); subtract the product (1) from the first position (leaving 4 in the dividend).
8. Multiply the quotient (1) times the second digit of the divisor (4); subtract the product (4) from the second position.
9. The answer (7) is now shown on the abacus.

To practice correcting under-estimates, work the following problems choosing the estimated quotient (EQ) indicated; then correcting to find the correct quotient (CQ).

	EQ	CQ		EQ	CQ
84 ÷ 21 =	3	4	99 ÷ 14 =	5	7 r1
96 ÷ 24 =	3	4	68 ÷ 13 =	3	5 r3
39 ÷ 13 =	2	3	46 ÷ 12 =	2	3 r10
90 ÷ 18 =	3	5	71 ÷ 11 =	4	6 r5
92 ÷ 23 =	2	4	54 ÷ 12 =	3	4 r6

	EQ	CQ		EQ	CQ
$74 \div 37 =$	1	2	$87 \div 17 =$	3	5 r2
$69 \div 23 =$	2	3	$39 \div 16 =$	1	2 r7
$57 \div 19 =$	2	3	$48 \div 13 =$	2	3 r9

Next we will consider problems where the estimated quotients are too high. Again we will use the problem $98 \div 14 = 7$, but this time our estimated quotient will be 8.

1, Set the divisor (14) on the columns farthest left.

2. Set the dividend (98) on the columns farthest right.

3. Divide the divisor (14) into the dividend (98); set the quotient (estimated to be 8) on the second column to the left.

4. Multiply the quotient (8) times the first digit of the divisor (1); subtract the product (8) from the first position (leaving 18 in the dividend).

5. Multiply the quotient (8) times the second digit of the divisor (4); subtract the product (32) from the second position (not possible).

It is at this point that we realize our estimated quotient is too high because the product of our last multiplication (32) cannot be subtracted from the remaining dividend (18). We then re-estimate the quotient to be 7, then:

6. Clear 1 from the estimated-quotient to obtain the re-estimation (7).

Because we had already multiplied the estimated quotient (8) times

the first digit of the divisor (1), and because we had already subtracted that product (8) from the first position, we must now correct the error by:

7. Multiply the number cleared from the estimated quotient (1) times whichever digits of the divisor were multiplied earlier by the incorrect estimated quotient (in this case, the first digit of the divisor (1) was the only one multiplied earlier). Add that product (1 x 1 = 1) to the first position.

Now we continue on with the division process where we left off when we discovered our error in estimating the quotient, i.e., after multiplying the quotient times the first digit of the divisor. Thus, we now:

8. Multiply the new quotient (7) times the second digit of the divisor (4). Subtract the product (28) from the second position.

9. The answer 7, is now shown on the abacus.

The example discussed above for correcting over-estimations should be studied carefully and worked several times so that the procedures are thoroughly memorized. Next work the following problems using the given estimated quotients first, and then use the procedure for correcting mis-estimates to obtain the correct quotient.

	EQ	CQ		EQ	CQ
90 ÷ 18 =	6	5	94 ÷ 18 =	7	5 r4
57 ÷ 19 =	4	3	98 ÷ 15 =	7	6 r8
85 ÷ 17 =	7	5	96 ÷ 13 =	8	7 r5
92 ÷ 16 =	7	5 r12	89 ÷ 17 =	6	5 r4
90 ÷ 15 =	8	6	98 ÷ 16 =	8	6 r2
91 ÷ 13 =	8	7	99 ÷ 19 =	7	5 r4
99 ÷ 14 =	8	7 r1	78 ÷ 19 =	5	4 r2

Now that you have learned how to correct both under and over-estimations, we can go on to consider larger division problems. There will be no more sets of exercises to practice corrections, but throughout the remainder of this unit on division you should continue to use the procedure for correcting mis-estimations whenever necessary.

Next we will consider problems having two-digit divisors and three digit dividends. Study the examples below and then do the practice exercises.

156 ÷ 39 = 4

1. Set the divisor (39) left.
2. Set the dividend (156) right.
3. Divide the divisor (39) into the dividend (156); set the quotient (4) on the column immediately to the left of the dividend. (When a two-digit divisor can not be divided into the first two digits of the dividend, the quotient is set on the column immediately to the left.)

4. Multiply the quotient (4) times the first digit of the divisor (3). Subtract the product (12) from the first position.
5. Multiply the quotient (4) times the second digit of the divisor (9). Subtract the product (36) from the second position.
6. The answer (4) is now shown on the abacus.

351 ÷ 27 = 13

1. Set the divisor (27) to the left.
2. Set the dividend (351) to the right.
3. Divide the divisor (27) into the first two digits of the dividend (35); set the quotient (1) in the second column to the left.
4. Multiply the quotient (1) times the first digit of the divisor (2). Subtract the product (2) from the first position.
5. Multiply the quotient (1) times the second digit of the divisor (7). Subtract the product (7) from the second position.
6. Divide the divisor (27) into the new dividend. (81). Set the quotient (3) on the second column to the left.
7. Multiply the quotient (3) times the first digit of the divisor (2). Subtract the product (6) from the first position.
8. Multiply the quotient (3) times the second digit of the divisor (7). Subtract the product (21) from the second position.
9. The answer (13) is now shown on the abacus.

658 ÷ 94 = 7	196 ÷ 14 = 14	387 ÷ 42 = 9 r9
148 ÷ 37 = 4	558 ÷ 31 = 18	485 ÷ 27 = 17 r26
268 ÷ 67 = 4	444 ÷ 12 = 37	987 ÷ 13 = 75 r12

568 ÷ 71 = 8	325 ÷ 25 = 13	612 ÷ 84 = 7 r24
282 ÷ 47 = 6	512 ÷ 16 = 32	872 ÷ 15 = 58 r2
292 ÷ 73 = 4	630 ÷ 42 = 15	763 ÷ 54 = 14 r7
168 ÷ 28 = 6	612 ÷ 12 = 51	295 ÷ 36 = 8 r7
364 ÷ 52 = 7	630 ÷ 10 = 63	345 ÷ 13 = 26 r7
195 ÷ 39 = 5	420 ÷ 42 = 10	574 ÷ 68 = 8 r30
320 ÷ 64 = 5	930 ÷ 31 = 30	786 ÷ 26 = 30 r6
414 ÷ 46 = 9	480 ÷ 24 = 20	732 ÷ 18 = 40 r12
225 ÷ 75 = 3	600 ÷ 12 = 50	694 ÷ 34 = 20 r14
150 ÷ 32 = 5	640 ÷ 16 = 40	278 ÷ 27 = 10 r8
672 ÷ 21 = 32	738 ÷ 64 = 11 r34	479 ÷ 16 = 29 r15

Next consider problems having a two-digit divisor and a dividend of four or more digits. Because the procedure for dividing does not change except to add more of the same steps, there will be no example. Work the problems below to solidify your understanding of division with two-digit divisors.

1092 ÷ 39 = 28	3160 ÷ 79 = 40	4503 ÷ 19 = 237
2714 ÷ 46 = 59	4270 ÷ 61 = 70	4872 ÷ 24 = 203
4956 ÷ 59 = 84	2700 ÷ 90 = 30	9246 ÷ 23 = 402
8245 ÷ 85 = 97	4000 ÷ 50 = 80	4912 ÷ 16 = 307
9801 ÷ 99 = 99	5778 ÷ 18 = 321	7106 ÷ 34 = 209
4914 ÷ 78 = 63	8435 ÷ 35 = 241	26901 ÷ 63 = 427
3192 ÷ 42 = 76	6552 ÷ 36 = 182	15562 ÷ 31 = 502

1825 ÷ 25 = 741	217728 ÷ 32 = 6804	679473 ÷ 48 = 14155 r33
98901 ÷ 99 = 999	195273 ÷ 39 = 5007	349682 ÷ 27 = 12951 r5
187461 ÷ 27 = 6943	153600 ÷ 50 = 3072	741319 ÷ 35 = 21180 r19
292373 ÷ 61 = 4793	260000 ÷ 40 = 6500	

Problems having three-or-more digit divisors are worked according to the same principles used thus far. Study the following example; then work the problem below.

5829 ÷ 374 = 15 r219

1. Set the divisor (374) to the far left.
2. Set the dividend (5829) to the far right.
3. Divide the divisor (374) into the first three digits of the dividend (582); set the quotient (1) in the second column to the left of the dividend. (Recall that if a three digit divisor can be divided into the first three digits of the dividend, the quotient is set on the second column to the left.)
4. Multiply the quotient (1) times the first digit of the divisor (3); subtract the product from the first position.
5. Multiply the quotient (1) times the second digit of the dividend (7); subtract the product (7) from the second position.
6. Multiply the quotient (1) times the third digit of the dividend (4); subtract the product (4) from the third position.

7. Divide the divisor (374) into the remaining dividend (208). Set the quotient (5) on the column immediately to the left. (If a three-digit divisor can not be divided into the first three digits of the dividend. The ouotient is set on the column immediately left.)
8. Multiply the quotient (5) times the first digit of the divisor (3). Subtract the product (15) from the first position.
9. Multiply the quotient (5) times the second digit of the divisor (7). Subtract the product (35) from the second position.
10. Multiply the quotient (5) times the third digit of the divisor (4). Subtract the product (20) from the third position.
11. The answer, (15 r219) is now shown on the abacus.

4508 ÷ 196 = 23	44,454 ÷ 239 = 186	71,042 ÷ 4287 = 16 r2450
8494 ÷ 274 = 31	165,282 ÷ 326 = 507	187,765 ÷ 1356 = 138 r637
5576 ÷ 328 = 17	158,762 ÷ 487 = 326	852,183 ÷ 2791 = 126 r517
10,056 ÷ 419 = 24	348,000 ÷ 580 = 600	918,673 ÷ 246 = 3734 r109
20,520 ÷ 684 = 30	285,640 ÷ 386 = 740	291,647 ÷ 683 = 427 r6
43,200 ÷ 720 = 60	47,894 ÷ 591 = 81 r23	824,792 ÷ 384 = 2147 r344
32,000 ÷ 400 = 80	67,432 ÷ 428 = 157 r236	825,864 ÷ 2971 = 277 r2897

ADDITIONAL PRACTICE

For additional practice in division, try these interesting problems.

2,222,222,202 ÷ 18 = 123,456,789

3,333,333,303 ÷ 27 = 123,456,789

4,444,444,404 ÷ 36 = 123,456,789

5,555,555,505 ÷ 45 = 123,456,789

6,666,666,606 ÷ 54 = 123,456,789

7,777,777,707 ÷ 63 = 123,456,789

8,888,888,808 ÷ 72 = 123,456,789

9,999,999,909 ÷ 81 = 123,456,789

DIVISION OF DECIMALS

If the number of decimal places in the dividend is greater than or equal to the number of decimal places in the divisor, then the procedure for division remains the same. We simply treat both numbers as whole numbers. Once the quotient is found, we subtract the number of decimal places in the divisor from the number of decimal places in the dividend; the difference indicates the number of decimal places that must be pointed off in the quotient. The example below illustrates this point.

5.69   4.7 = 1.2 r5

1. Set the divisor (47) to the far left.
2. Set the dividend (569) to the far right.
3. Divide the divisor (47), into the first two digits of the dividend (56); set the quotient (1) on the second column to the left.
4. Multiply the quotient (1) times the first digit of the divisor (4); subtract the product (4) from the first position.
5. Multiply the quotient (1) times the second digit of the divisor (7); subtract the product (7) from the second position.

6. Divide the divisor (47) into the first two digits of the remaining dividend (99) set the quotient (2) on the second column to the left.
7. Multiply the quotient (2) times the first digit of the divisor (4); subtract the product (8) from the first position.
8. Multiply the quotient (2) times the second digit of the divisor (7); subtract the product (14) from the second position.
9. The answer 12 r5 is now shown on the abacus: dividend had two decimal places. We subtract the one decimal place in the divisor from the two decimal places in the dividend and find we must point off one decimal place in the quotient - thus, our final answer is 1.2 r5.

If, however, there are fewer decimal places in the dividend than in the divisor, then we must add as many zeroes to the dividend as there are fewer decimal places. We then divide as usual and, since we now have an equal number of decimal places, in both dividend and divisor. There are no places to be pointed off in the quotient. Study the example below.

$791 \div 3.5 = 226$

Because the dividend has one fewer decimal places than the divisor, we must add that many (1) zeroes to it before setting. Thus when we consider the divisor and dividend as whole numbers, we have 35 and 7910 respectively.

1. Set the divisor (35) to the far left.
2. Set the dividend (7910) to the far right.

3. Divide the divisor (35) into the first two digits of the dividend (79). Set the quotient (2) on the second column to the left.

4. Multiply the quotient (2) times the first digit of the divisor (3). Subtract the product (6) from the first position.

5 Multiply the quotient (2) times the second digit of the divisor (5). Subtract the product (10) from the second position.

6. Divide the divisor (35) into the first two digits of the remaining dividend (91). Set the quotient (2) on the second column to the left.

7. Multiply the quotient (2) times the first digit of the divisor (3). Subtract the product (6) from the first position.

8. Multiply the quotient (2) times the second digit of the divisor (5). Subtract the product (10) from the second position.

9. Divide the divisor (35) into the first three digits of the remaining dividend (210); set the quotient (6) on the column immediately to the left.

10. Multiply the quotient (6) times the first digit of the divisor (3). Subtract the product (18) from the first position.

11. Multiply the quotient (6) times the second digit of the divisor (5). Subtract the product (30) from the second position.

12. The answer (226) is now shown on the abacus. Because we added

zeroes to the dividend until the number of decimal places in the dividend was equal to the number in the divisor, there is no difference between the two, and thus, no decimal places to be pointed off in the answer.

Work these problems for practice in dividing with decimals.

35.67 ÷ 6.8 = 5.2 r31	84 ÷ 6 = 840 ÷ 6 = 140
6.59 ÷ 4.7 = 1.4 r1	48 ÷ 7 = 480 ÷ 7 = 68 r4
78.4 ÷ .8 = 98	26.8 ÷ .49 = 26.80 ÷ 49 = 54 r34
8.996 ÷ 3.91 = 2.3 r3	68 ÷ .28 = 6800 ÷ 28 = 242 r24
38.6 ÷ .42 = 91 r38	70 ÷ .321 = 70000 ÷ 321 = 218 r22

FRACTIONS

In a fraction, such as 2/5, 8/3, 7/10, 4/4 etc., we know that the number below the line, the denominator, indicates how many equal parts a whole is divided into; and we know that the number above the line, the numerator, indicates how many of these equal parts we are to consider. When a number includes both a whole number and a fraction, it is called a mixed number.

Fractions and mixed numbers can be added, subtracted, multiplied, and/or divided as can whole numbers. Sometimes it is necessary to reduce fractions such as 4/8, 6/9, 12/48 or 15/15 which have not been written in their lowest terms.

When reducing, adding, or subtracting fractions or mixed numbers, the abacus is divided into three sections - one each for the denominator, numerator, and whole number. Beginning at the extreme right of the abacus, the first three columns are the section for the denominator. The denominator is always written as far right in the section as possible: thus a one-digit denominator (3 or 9) is always written on the first column from the right, a two-digit denominator (48 or 15) is always written on columns 1 and 2 from the right, and a three digit denominator is written on columns 1, 2, and 3 from the right.

The next three columns (columns 4, 5, and 6 from the right) are the section for the numerator. The numerator, like the denominator, is set as far right within its own section as is possible: thus a one-digit numerator is always set on column 4 from the right.

The next three columns (columns 7, 8, and 9 from the right) are the section for the whole number. The whole number, too, is set as far right within its own section as is possible. Notice then that each section is set off by the comma markers, and that the last column of each section is easily identifiable - a feature that is especially important when any part of a mixed number ends with a zero.

REDUCTION OF FRACTIONS

We will first consider the reduction of a fraction to its lowest term. When the numerator is smaller than the denominator, we proceed as follows.

6/9

1. Set the numerator (6) as far right in the numerator section as possible. (i.e. column 4 from the right).
2. Set the denominator (9) as far right in the denominator section as possible (i.e. column 1 from the right).
3. Divide the numerator (6) by the lowest common factor of both numerator and denominator (3); clear the numerator (6) and set the quotient (2) in its place.
4. Divide the denominator (9) by the lowest common factor (3); clear the denominator (9) and set the quotient (3) in its place.
5. The answer (2/3) is now shown on the abacus. (We know that the numerator is 2, and not 20, or 200, because the 2 is set on the column farthest right within the numerator section).

With larger fractions, such as 12/64, we may first choose a common factor to divide by which is not the lowest possible, such as 2. If so, our resulting fraction 6/32 will still not be in lowest terms. Thus we proceed by choosing a common factor of the resulting fraction 6/32, and repeat the reduction process.

Reduce these fractions to lowest terms.

4/8 = 1/2	9/27 = 1/3	16/48 = 1/3	17/85 = 1/5
2/10 = 1/5	7/63 = 1/9	11/88 = 1/8	15/60 = 1/4
5/20 = 1/4	3/48 = 1/16	18/162 = 1/9	26/182 = 1/7
6/36 = 1/6	5/95 = 1/19	12/132 = 1/11	8/12 = 2/3
12/16 = 3/4	18/27 = 2/3	8/10 = 4/5	15/40 = 3/8
56/64 = 7/8	6/8 = 3/4	28/35 = 4/5	45/72 = 5/8

If the numerator of a fraction is not smaller than the denominator, we follow the same procedure for reducing it to lowest terms, but then we must continue working to change it to a whole or mixed number. For instance, if we have the fraction 18/8, we reduce it to 9/4 by using the procedure already discussed. Having done so, we should have 9 set in the numerator section and 4 set in the denominator section. We then change the fraction to a mixed number by:

1. Divide the denominator (4) into the numerator (9); set the quotient (2) in the section for whole numbers, as far right in the section as is possible - (i.e. on column 7 from the right).
2. Multiply the quotient (2) times the denominator (4), Subtract the product (8) from the numerator (9)

3. The answer, 2 1/4, is now shown on the abacus. (We know that the 2 is not 20, or 200 and that the one is not 10 or 100 because both numbers are written as far right within their own sections as possible.)

Reduce these fractions, then change them to whole or mixed numbers.

8/6 = 4/3 = 1 1/3	28/16 = 7/4 = 1 3/4
6/2 = 3/1 = 3	55/40 = 11/8 = 1 3/8
25/5 = 5/1 = 5	72/42 = 12/7 = 1 5/7
28/5 = 5 3/5	60/25 = 12/5 = 2 2/5
14/4 = 7/2 = 3 1/2	84/16 = 21/4 = 5 1/4
18/16 = 9/8 = 8 1/8	45/36 = 5/4 = 1 1/4
12/10 = 6/5 = 1 1/5	95/75 = 19/15 = 1 4/15

ADDITION OF FRACTIONS

Before we can begin to add any two (or more) fractions together, we must first choose a common denominator, preferably the lowest. Each fraction to be added, must first be changed to a fraction of equal value having that common denominator. The following example will show how this is done.

4 3/5 + 5 2/7 = 9 31/35

1. Set the common denominator (35) in the denominator section.
2. Set the whole number of the first number to be added (4) in the whole number section.

We must now change the 3/5 to a fraction of equal value having the common denominator. To do so:

3. Set the numerator of the first fraction (3) on the far left column of the abacus.
4. Skip a column and on the next one set the denominator of the first fraction (5).
5. Divide the denominator of the first fraction (5) into the common denominator (35); then multiply that quotient (7) times the numerator of the first fraction (3); set the product (21) in the numerator section.
6. Clear the first fraction (3/5) from the left columns of the abacus.
7. Add the whole number of the second number (5) to whatever is already set in the whole number section (4).
8. Set the numerator of the second fraction (2) on the far left column of the abacus.
9. Skip a column and set the denominator of the second fraction (7).
10. Divide the denominator of the second fraction (7) into the common denominator (35); then multiply that quotient (5) times the numerator of the second fraction (2); and the product (10) to whatever is already set in the numerator section. (21)
11. Clear the second fraction (2/7) from the left columns of the abacus.
12. The answer (9 31/35) is now shown on the abacus.

If, after adding fractions, the numerator of the fraction in the answer is larger than the denominator as in 6 9/7, then the fraction must be changed to a mixed number (1 2/7) in the manner discussed earlier, and the whole number part is added to whatever is already set in the whole number section - thus we would have 7 2/7.

Add these fractions for practice.

1/5 + 3/5 = 4/5	7/8 + 2/3 = 37/24 = 1 13/24
1/3 + 1/4 = 7/12	6 2/9 + 4 5/6 = 10 19/18 = 11 1/18
3/5 + 1/6 = 23/30	6 5/8 + 7 7/12 = 13 29/24 = 14 5/24
1/3 + 3/8 = 17/24	3 13/16 + 7 11/32 = 10 37/32 = 11 5/32
4 2/7 + 4/7 = 4 6/7	2 5/7 + 4 12/21 = 6 27/21 = 7 6/21 = 7 2/7
7 3/8 + 2 5/24 = 9 7/12	7 11/18 + 19 5/6 = 16 26/18 = 17 8/18 = 17 4/9
3/4 + 2/3 = 17/12 = 1 5/12	

SUBTRACTION OF FRACTIONS

The procedure for subtracting fractions is exactly the same as for adding them except that, in addition, we added the second whole number and numerator to the first but in subtraction we subtract the second whole number and numerator from the first. It sometimes happens that the numerator of the first fraction is smaller than the second; and thus, the second can not be subtracted from it. When this occurs, we borrow 1 from the first whole number and change it to a fraction having whatever common denominator we have chosen. For example, if our common denominator is 4, we borrow 1 and change it to 4/4 since 4/4 equals 1. If the common denominator is 18, we change 1 to 18/18.

Next we add the numerator of the fraction changed from 1 to the numerator of the first fraction, and then proceed with our subtraction of the second fraction. (Notice that we borrowed 1 from the first whole number and added it in fraction form to the first numerator. Thus we did not change the value of the first mixed number, only the form.)

Study the subtraction example below.

7 2/9 - 1 5/6 = 5 7/18

1. Set the common denominator (18) in the denominator section.
2. Set the whole number of the first number (7) in the whole number section.
3. Set the numerator of the first fraction (2) on the far left column of the abacus.
4. Skip a column and on the next one set the denominator of the first fraction. (9)
5. Divide the denominator of the first fraction (9) into the common denominator (18); then multiply that quotient (2) times the numerator of the first fraction (2); set the product (4) in the numerator section.
6. Clear the first fraction (2/9) from the left columns of the abacus.
7. Subtract the whole number of the second number (1) from the whole number of the first. (7)
8. Set the numerator of the second fraction (5) on the far left column of the abacus.

9. Skip one column, and on the next one set the denominator of the second fraction (6).
10. Divide the denominator of the second fraction (6) into the common denominator (18); multiply that quotient (3) times the numerator of the second fraction (5); the product should now be subtracted from the numerator of the first fraction, but in this case the first numerator is smaller, Thus
11. Borrow 1 from the whole number section by clearing it and changing it to a fraction with the common denominator (18/18) add the numerator to the numerator section.
12. Now subtract the numerator of the second fraction as found in step 10 (15).
13. The answer (5 7/18) is now shown on the abacus.

Work the following problems for practice in subtraction.

5/7 - 3/7 = 2/7

2 5/8 - 1 2/8 = 1 1/4

4/5 - 2/15 = 10/15 = 2/3

5 2/3 - 3 1/7 = 2 11/21

18 5/6 - 8 3/5 = 10 7/30

1 1/3 - 3/5 = 11/15

8 1/4 - 6 5/12 = 1 10/12 = 1 5/6

9 2/7 - 3 9/14 = 5 9/14

21 5/12 - 17 31/36 = 3 20/36 = 3 5/9

18 4/13 - 12 64/65 = 5 21/65

29 4/5 - 16 4/5 = 12 13/45

17 3/29 - 2 51/58 = 14 13/58

6 3/10 - 2 3/4 = 3 22/40 = 3 11/20

MULTIPLICATION OF FRACTIONS

At the beginning of this chapter on fractions we noted that when reducing, adding, or subtracting either fractions or mixed numbers, we divide the abacus into three sections. Now we are ready to consider multiplication and division of fractions and mixed numbers; and when this is done, the abacus is divided into two sections, one being the left side for the numerators and the other being the right side for the denominators. The numerators are set as far left as possible with one empty column separating them. The denominators are set as far right as possible, again with one vacant column separating them. And as you know, to multiply two fractions we multiply one numerator times the other and the product becomes the numerator of the answer. Then we multiply one denominator times the other and the product becomes the denominator of the answer. Sometimes, of course, the answer must be reduced.

Study the following example. Remember to set and clear numbers with the proper fingers.

4/7 x 5/9 = 20/63

1. Set the numerator of the first fraction (4) on the column farthest left.
2. Set the denominator of the first fraction (7) on the column farthest right.
3. Leave one column vacant to the right of the first numerator, and set the numerator of the second fraction (5) on the next column to the right.
4. Leave one column vacant to the left of the first denominator

and set the denominator of the second denominator (9) on the next column to the left.

5. Now multiply the first numerator (4) times the second (5), clear the numerators, then set the product (20) so that the last digit is on the fourth column from the left edge of the abacus. Notice that there is a comma marker just to the right of this column - thus the position of the last digit is easily identifiable and numbers ending in zero should not be misread.
6. Next multiply the first denominator (7) times the second (9); clear both denominators, then set the product (63) so that the last digit is on the column farthest right.
7. The answer, (20/63) is now shown on the abacus.

Multiply these fractions for practice. Reduce the answers if necessary.

5/6 x 3/4 = 15/24 = 5/8	3/5 x 1/3 = 3/15 = 1/5
1/3 x 2/3 = 2/9	7/9 x 6/7 = 42/63 = 6/9 = 2/3
7/8 x 2/5 = 14/40 = 7/20	1/12 x 1/4 = 1/48
4/7 x 9/10 = 36/70 = 18/35	4/5 x 4/9 = 16/45
5/12 x 1/2 = 5/24	1/6 x 3/8 = 3/48
4/15 x 2/3 = 8/45	5/11 x 2/3 = 10/33

We have noted that in multiplication and division, there is no section for whole numbers. However, we are still able to multiply mixed numbers on the abacus. To do so we simply change the mixed

numbers to improper fractions then set and work them as we did the fractions in the last group of exercises. Do these problems for practice. Reduce and change the answers to mixed numbers if necessary.

1 1/3 x 4 2/5 = 4/2 x 22/5 = 88/15 = 5 13/15

2 2/3 x 1 1/4 = 8/3 x 5/4 = 40/12 = 10/3 = 3 1/3

5/9 x 5 1/2 = 5/9 x 11/2 = 55/18 = 3 1/18

4 2/3 x 2 = 14/3 x 2/1 = 28/3 = 9 1/3

2 1/6 x 1 1/4 = 13/6 x 5/4 = 65/24 = 2 17/24

CANCELLATION

One other process sometimes used in multiplication (and division) is the process of cancellation, i.e. when some number can be divided evenly into one of the numerators and one of the denominators.

Study this example to see how this process is done on the abacus.

1 5/7 x 2/3 = 12/7 x 2/3 = 4/7 x 2/1 = 8/7 = 1 1/7

1. Set the numerator of the first fraction (12).
2. Set the denominator of the first fraction (7).
3. Set the numerator of the second fraction (2).
4. Set the denominator of the second fraction (3).
5. Place the left hand on the first numerator (12). With the right hand check the left-most denominator (3). We find that the number 3 can be divided evenly into both. Thus

6. Divide 3 into the first numerator (12); with the right hand clear the first numerator (12) and set the quotient (4) in its place.
7. Divide the 3 into the left-most denominator, clear that denominator and set the quotient (1) in its place.
8. Now with the left hand again on the first numerator (4), check the far right denominator (7); there is no number that will cancel into them evenly.
9. Next place the left hand on the second numerator (2); with the right hand check the far-left denominator (1), again there is no number that will cancel.
10. With the left hand still on the second numerator (2), check the far-right denominator (7); once again there is no number that can be divided evenly into both numbers.

Now we have completed cancellation. From this point, the multiplication procedure continues as before. Work these problems for practice. Be sure to cancel whenever possible and to reduce the final answers and change them to mixed numbers when necessary.

3/4 x 8/11 = 3/1 x 2/11 = 6/11

1/9 x 3/5 = 1/3 x 1/5 = 1/15

4/7 x 1/12 = 1/7 x 1/3 = 1/21

1 3/7 x 5/6 = 10/7 x 5/6 = 5/7 x 5/3 = 25/21 = 1 4/21

1 7/8 x 1 1/3 = 15/8 x 4/3 = 5/2 x 1/1 = 5/2 = 2 1/2

1 4/5 x 2 1/12 = 9/5 x 25/12 = 2/1 x 5/4 = 15/4 = 3 3/4

2 5/8 x 1 5/7 = 21/8 x 12/7 = 3/2 x 3/1 = 9/2 = 4 1/2

1 7/9 x 2 7/10 = 16/9 x 27/10 = 8/1 x 3/5 = 24/5 = 4 4/5

DIVISION OF FRACTIONS

To divide one fraction by another, we simply invert the divisor, and multiply in the manner already learned. In other words, in the problem $2/3 \div 1/5 =$ we simply invert the divisor and multiply. i.e. $2/3 \times 5/1 = 10/3 = 3\ 1/3$. As in multiplication, we must sometimes change mixed numbers to improper fractions. We are oftentimes able to cancel, and we frequently find it necessary to reduce the answers and/or change them to mixed numbers. Since we have already dealt with all of these processes when considering multiplication, we will not discuss them again here. However, you will need to use them in solving the following problems:

$4/5 \div 3/7 = 4/5 \times 7/3 = 28/15 = 1\ 13/15$

$5/8 \div 1/3 = 5/8 \times 3/1 = 15/8 = 1\ 7/8$

$7/9 \div 2/5 = 7/9 \times 5/2 = 35/18 = 1\ 17/18$

$3/4 \div 5/6 = 3/4 \times 6/5 = 18/20 = 9/10$

$2/3 \div 5/7 = 2/3 \times 7/5 = 14/15$

$1\ 5/7 \div 4\ 1/2 = 12/7 \div 9/2 = 12/7 \times 2/9 = 24/63 = 8/21$

$1\ 7/8 \div 1\ 1/2 = 15/8 \div 3/2 = 15/8 \times 2/3 = 5/4 \times 1/1 = 5/4 = 1\ 1/4$

$1\ 5/12 \div 2\ 5/6 = 17/12 \div 17/6 = 17/12 \times 6/17 = 1/2 \times 1/1 = 1/2$

$3\ 1/8 \div 1\ 1/4 = 25/8 \div 5/4 = 25/8 \times 4/5 = 5/2 \times 1/1 = 5/2 = 2\ 1/2$

$3\ 1/9 \div 1\ 11/21 = 28/9 \div 32/21 = 28/9 \times 21/32 = 7/3 \times 7/8 = 49/24 = 2\ 1/24$

$3\ 3/5 \div 1\ 6/15 = 18/5 \div 21/15 = 18/5 \times 15/21 = 6/1 \times 3/7 = 18/7 = 2\ 4/7$

EXTRACTION OF ROOTS

To describe the complete process for extracting roots would necessitate considerable arithmetical instruction as well as a good deal of space, because the computations are rather involved and will likely require the use of more than one abacus, I feel it is impractical to explain the procedure here.

If you have available a table of logarithms, extracting roots becomes much simpler. Study the example below to learn exactly how such extractions are done.

To find the square root of 144

1. Find the logarithm (1584) of the number whose root you want to extract (144).
2. Divide that logarithm (1584) by the number of the root (2).
3. Read the quotient (792); find that quotient in the logarithm table; it will be the logarithm of the answer you are seeking (12).

With the aid of a logarithm table, extract the requested roots from these numbers.

square root of 121 = 11	cube root of 1331 = 11
square root of 196 = 14	cube root of 2197 = 13
square root of 100 = 10	cube root of 1000 = 10
square root of 169 = 13	cube root of 1728 = 12
square root of 184 = 13.565	fourth root of 10000 = 10
square root of 137 = 11.705	fourth root of 38416 = 14
square root of 154 = 12.410	

Tables of logarithms are available in Braille from the American Printing House for the Blind, 1839 Frankfort Avenue, Louisville, 6, Kentucky.

CONCLUSION

On the preceding pages we have discussed the arithmetical process of addition, multiplication, subtraction, and division, as well as work with fractions and the extraction of roots. Naturally, the mathematical examples and exercises in this book are quite elementary since the purpose was to teach you how to use the abacus and not how to solve problems of higher math. Blind students who learn to use the abacus should not, however, assume that their abacus can be used to solve only the simplest problems of addition, subtraction, multiplication, or division. It should be obvious that these elementary processes are, in fact, used in all types of higher mathematics from geometry to calculus and that wherever these processes are employed, the abacus will continue to be useful. Of course, the braillewriter or slate and stylus will be necessary to record intermediate answers and final solutions, but the abacus will remain the quickest and most effective means for finding those answers.

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